RESEARCH STATEMENT

JACK SHOTTON

Introduction

I am interested in the interplay of congruences between automorphic forms, moduli spaces of Galois representations, and modular representation theory of p-adic Lie groups.

Before discussing background, I will briefly summarise my work to date. Let l and p be distinct primes, F a p-adic field, and $\mathcal{O}$ its ring of integers. My research achievements are:

A Explicitly computing deformation rings of two-dimensional mod l representations of G_F, and answering questions about their geometry. See [Sho16] and [Sho17b].

B Formulating and proving a relation between, on the one hand, moduli spaces of n-dimensional mod l representations of G_F and, on the other hand, the mod l representation theory of $GL_n(\mathcal{O})$. This relation is an analogue, when $l \neq p$, of the (very much open) Breuil–Mézard conjecture, and puts on a systematic footing the "Ihara avoidance" method of [Tay08]. See [Sho16], [Sho17a], and [Sho].

C In work in progress with Jeffrey Manning, proving new cases of Ihara's lemma for the $\mathbb{F}_l$-cohomology of Shimura curves over $\mathbb{Q}$, using a method that we hope will generalise to Shimura curves totally real fields. See [MS].

D Proving a basic finiteness result in the mod p representation theory of $GL_2(F)$ (unlike the other results discussed, this is a result about congruences mod p of local representations at the same prime p. Thus it fits into the mod p/p-adic Langlands program). See [Sho17c].

The objects of study in my research are automorphic forms and Galois representations. Automorphic forms come from the spectral theory of Lie groups, and the first examples that one meets are the modular forms. If $\Gamma \subset SL_2(\mathbb{Z})$ is a subgroup defined by congruences, and $k \geq 2$ is an integer, then a modular form of level Γ and weight k is a holomorphic function f on the complex upper half plane which satisfies the transformation law

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. The finite-dimensional vector space of modular forms of a given weight and level carries a family of commuting operators, the Hecke operators T_p defined for almost all primes p. We are primarily interested in their simultaneous eigenvectors, or *eigenforms*.

Many Galois representations come from varieties defined over $\mathbb{Q}$. If E is an elliptic curve over $\mathbb{Q}$, then, for every prime l, the l-torsion $E[l]$ has an action of the absolute Galois group of $\mathbb{Q}$. These actions know about the arithmetic of the elliptic curve; in particular, for almost every prime p, they know the number n_p of mod p solutions of the equation defining E. It is a remarkable theorem (see [Wil95] and [BCDT01]) that, for every elliptic curve over $\mathbb{Q}$, there is a weight two eigenform such that the eigenvalue of T_p is $1 + p - n_p$ for all but finitely many primes p.

Theorems like this are proved by studying congruences between modular forms and between Galois representations, which is my area of research. For more detailed background, see the next section. The subsequent sections discuss the results listed as **A** to **D** above and the further work that they suggest.

Background

By congruences between automorphic forms, I mean phenomena such as the following. Let f be the unique normalised cuspidal modular form of weight 6 and level $\Gamma_0(3)$, the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices whose bottom-left coefficient is a multiple of 3. It has Fourier expansion in terms of $q = e^{2\pi i z}$ of the form

$$\begin{aligned} f(z) &= \sum_{n \geq 1} a_n q^n \\ &= q - 6q^2 + 9q^3 + 4q^4 + 6q^5 - 54q^6 - 40q^7 + 168q^8 + 81q^9 + \dots . \end{aligned}$$

A level raising theorem[1] predicts that, if $l \neq 3, 13$ and $p \neq 3$ are distinct primes, and

$$a_p \equiv \pm p^2(1 + p) \mod l, \tag{1}$$

then there is a modular form g of weight 6 and level $\Gamma_0(3p)$ whose Fourier coefficients b_n for $p \nmid n$ are congruent to those of f modulo l. And indeed, we see that this happens for $p = 7$ and $l = 11$:

$$a_7 = -40 \equiv 7^2(1 + 7) \mod 11$$

and there is an eigenform g of weight 6 and level $\Gamma_0(21)$ with q-expansion

$$q + 5q^2 + 9q^3 - 7q^4 + 94q^5 + 45q^6 - 49q^7 - 195q^8 + 81q^9 + \dots .$$

[1] In this case a theorem of Diamond [Dia91] generalising work of Ribet [Rib84].

What is going on here? To understand such examples systematically, it is necessary to think in terms of representations, of two different sorts. The modular form f generates an *automorphic* representation; roughly, this is a tensor product (over all p) of representations π_p of $GL_2(\mathbb{Q}_p)$, but satisfying a very strong *global* condition linking the actions of the *local* groups $GL_2(\mathbb{Q}_p)$ at the varying p. These local representations contain the data of the a_p (they are simply the eigenvalues of the Hecke operators). But associated to f is also a *Galois* representation ρ_f, a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This restricts to give *local* Galois representations $\rho_{f,p}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for every p, which also contain the data of the a_p. That these local Galois representations came from a representation of $G_{\mathbb{Q}}$ is again a strong *global* condition. The point of the Langlands program from the point of view of a number theorist is that 1) the local data on the Galois and automorphic sides can be matched up ("local Langlands") and that 2) the global conditions that arise on both sides also match up. Proving results like this can result in great theorems — the truth of Fermat's last theorem, the analytic continuation of L-functions of elliptic curves over $\mathbb{Q}$, the Sato–Tate conjecture, ...

Now, in the setup above, the existence of g is a global question that imposes local constraints — this is the source of the condition (1) on a_p. That the local constraints are the same whether viewed from the Galois or automorphic point of view is a sort of compatibility of the local Langlands correspondence with reduction modulo l. The content of the level raising theorem is that the necessary local constraints are actually sufficient.

There are (at least) two approaches to proving level raising results. The first, due to Ribet [Rib84], uses a result known as "Ihara's lemma" [Iha75] about the mod l cohomology of modular curves. For automorphic forms of higher rank a conjectural generalisation was formulated in [CHT08], and is quite open. The second, due to Taylor [Tay08], uses a method called "patching". It uses the geometry of moduli spaces of local Galois representations. In the end the patching method has proved the most general level raising theorems, but Ihara's lemma gives more detailed information.

Patching is a heavy part of much of my work, but it is rather technical. Let me just say that, from a modern point of view, it is a way of taking limits of (finite dimensional!) spaces of automorphic forms to obtain coherent sheaves over moduli spaces of local Galois representations. These sheaves are maximal Cohen–Macaulay. This property, and the geometry of the moduli spaces, can then be used to prove results about the original spaces of automorphic forms. For instance, the multiplicity one results of [Dia97], the proof of the Fontaine–Mazur conjecture in [Kis09], and the non-minimal automorphy lifting theorems of [Tay08] all use this perspective.

A. Moduli of local Galois representations when $\ell \neq p$.

Suppose that F is a finite extension of $\mathbb{Q}_p$, and let $G_F = \mathrm{Gal}(\overline{F}/F)$. We would like to make sense of the *moduli space of representations* $\mathcal{M}$ of G_F over $\mathbb{Z}_l$. There are two cases:

- if $l = p$, such a moduli space of *lattices in crystalline representations* was constructed recently by Emerton and Gee (see [EG15] and forthcoming work) using integral p-adic Hodge theory.
- if $l \neq p$, the restriction of a family of $\mathbb{Z}_l$-representations to the pro-p *wild* inertia subgroup is locally constant, and so we can 'factor it out'. So it is enough to consider the moduli of representations of the tame quotient of G_F, which is isomorphic to the profinite completion of the group
$$\left\langle \Sigma, \Phi : \Phi\Sigma\Phi^{-1} = \Sigma^q \right\rangle$$
where q is the order of the residue field of F. Thus we consider the affine scheme
$$\mathcal{M} = \left\{(\Phi, \Sigma) \in (GL_n)_{\mathbb{Z}_l} : \Phi\Sigma\Phi^{-1} = \Sigma^q\right\}$$
which parametrises tamely ramified representations of G_F.[2]

I study the geometry of $\mathcal{M}$, both for its intrinsic interest as a moduli space and for possible global applications via patching. When $n = 2$, I studied $\mathcal{M}$ very explicitly in [Sho16] (for $l > 2$) and [Sho17b] (for $l = 2$). In particular I could obtain explicit equations for all of its geometrically irreducible components, and prove results about their geometry. For instance, to every degree n polynomial τ, we can associate a closed subscheme $\mathcal{M}^\tau$ of $\mathcal{M}_{\overline{\mathbb{Z}}_l}$ formed as the union of those geometrically irreducible components on which the characteristic polynomial of Σ is τ.

Theorem A.1. *When $n = 2$, for any degree two polynomial τ, the scheme $\mathcal{M}^\tau$ is Cohen–Macaulay.*

It was previously proved by Helm that the *whole* space $\mathcal{M}$ is a complete intersection, and so Cohen–Macaulay; by contrast, the subschemes $\mathcal{M}^\tau$ are not always Gorenstein (and so not complete intersections). Theorem A.1 was used in [HP17] to show that certain deformation rings for two-dimensional *global* Galois representations are torsion-free. The explicit calculations behind it have been applied by Manning [Man] to compute the multiplicity of mod l Galois representations in the cohomology of Shimura curves (unlike in the modular curve case, the multiplicity is not 1!)

For higher n, I will consider the following question:

[2]The restriction to the tame case is made only for simplicity; in general the moduli spaces will be products of spaces such as $\mathcal{M}$.

Question A.2. *Is Theorem A.1 true when $n > 2$?*

B. The Breuil–Mézard conjecture when $l \neq p$

The local Langlands correspondence (for GL_n/F) is a bijection "rec" between

$$\{\text{certain representations of } GL_n(F)\}$$

and

$$\{\text{certain } n\text{-dimensional representations of } G_F\}.$$

That this can be interpolated over the moduli space $\mathcal{M}$ of $\mathbb{Z}_l$-representations of G_F was a conjecture of Emerton and Helm [EH11], now proved by Helm and Moss [HM16].

The inertial Langlands correspondence, a consequence of type theory, says that the restriction of $\mathrm{rec}(\pi)$ to inertia can be read off from whether π contains certain representations of a maximal compact subgroup $K = GL_n(\mathcal{O}) \subset GL_n(F)$ (see [BK93], [BK99] and [SZ99]). The components of $\mathcal{M}$ are in bijection with the possible restrictions of a Galois representation to inertia. Thus we have a connection between representations of K and components of $\mathcal{M}$. More precisely, in [Sho17a] I defined an explicit way to associate, to every characteristic zero representation σ of K, a formal sum $\mathcal{C}(\sigma)$ of irreducible components of $\mathcal{M}$. This is compatible with reduction modulo l in a precise sense:

Theorem B.1. *It is possible to associate, to every mod l representation σ of K, a formal sum $\overline{\mathcal{C}}(\sigma)$ of irreducible components of the special fibre of $\mathcal{M}$, such that $\mathcal{C}$ and $\overline{\mathcal{C}}$ are compatible with reduction modulo l.*

Reduction modulo l of a K-representation σ takes place in the Grothendieck group; reduction modulo l of a component of $\mathcal{M}$ means taking its intersection, with multiplicities, with the special fibre. When $l = p$, a similar statement had been formulated in [BM02] (for GL_2) and [EG14] (for GL_n), in which case there are many more congruences on both sides and this conjecture — the Breuil–Mézard conjecture — is very much open. When $n = 2$ many cases were proved in [Kis09], with application to the Fontaine–Mazur conjecture.

In the special case that $q \equiv 1 \mod l$ and $l > n$, the congruence relations between components that the theorem predicts are those that were used in Taylor's proof of non-minimal automorphy lifting for automorphic forms on definite unitary groups. One purpose of the theorem is to put these congruences on a systematic footing.

The proof of Theorem B.1 in [Sho17a] used patching to provide an exact functor from representations of K to maximal Cohen–Macaulay sheaves over $\mathcal{M}$, thus connecting the two sides of the story. This uses a global argument to prove a local statement; I have since found a purely local proof in the tame case, which also works when $l = 2$. This will be the content of [Sho]. It relies on the combinatorics of symmetric functions and on the modular representation theory of finite general linear groups.

Problem B.2. *Generalise Theorem B.1 to connected reductive groups G other than GL_n.*

There has been recent work [BG17] [BP17] on the moduli spaces of local Galois representations for such G, but it all requires first inverting l. Part of generalising these results will involve understanding the basic geometry of these moduli spaces over $\mathbb{Z}_l$. I will investigate this with both local and the global methods.

C. Ihara's lemma for Shimura curves

Modular curves are quotients $X_\Gamma = \Gamma\backslash\mathcal{H}$, where $\mathcal{H} = \mathbb{C} - \mathbb{R}$ and Γ is a congruence subgroup of $GL_2(\mathbb{Z})$; they are non-compact Riemann surfaces (which can be defined over $\mathbb{Q}$). If instead we take Γ to be a congruence subgroup of $\mathcal{O}_D$, the units of an order in an indefinite quaternion algebra D over $\mathbb{Q}$, then we obtain a **Shimura curve**, a compact Riemann surface. Ihara's lemma concerns the mod l cohomology $H^1(X_\Gamma, \mathbb{F}_l)$ of either kind of curve.

Fix such a D and Γ. If p is a prime not dividing l, the level of Γ, or the discriminant of D, then we can form the subgroup $\Gamma' = \Gamma_0(p)$ of elements that are upper triangular mod p (where we choose an isomorphism $\mathcal{O}_D \otimes \mathbb{Z}_p \xrightarrow{\sim} M_2(\mathbb{Z}_p)$). There are two natural maps

$$\pi_1, \pi_2 : X_{\Gamma'} \to X_\Gamma.$$

If l is a prime distinct from p, then we can consider the induced maps

$$\pi_i^* : H^1(X_{\Gamma'}, \mathbb{F}_l) \to H^1(X_\Gamma, \mathbb{F}_l).$$

Let $\pi^* : H^1(X_{\Gamma'}, \mathbb{F}_l)^{\oplus 2} \to H^1(X_\Gamma, \mathbb{F}_l)$ be the sum of these maps. Then π^* is not injective, but one has in general (see [CT11]):

Conjecture C.1. *(Ihara's lemma for Shimura curves) The kernel of π^* is Eisenstein.*

"Eisenstein" means the following. All these cohomology groups have an action of the ring $\mathbb{T}$ of Hecke operators, and are supported at some finite set of maximal ideals of $\mathbb{T}$. Each such maximal ideal $\mathfrak{m}$ has an associated representation $\rho_\mathfrak{m}$ of $G_\mathbb{Q}$ landing in $\mathbb{T}/\mathfrak{m}$. To say that the kernel of π^* is Eisenstein means that, for every maximal ideal in its support, the associated Galois representation is reducible.

This conjecture is known for modular curves by work of Ribet [Rib84], and for Shimura curves if $l > 3$ and l does not divide the discriminant of D or the level Γ by [DT94]. It was used in these works to prove level raising results for modular forms: if f is a modular form of weight two and level Γ such that $a_p(f) \equiv \pm(1+p) \mod l$, then f is congruent to a modular form of weight two and level Γ' that is new at p. These days there are other ways of proving such results but the conjecture gives finer information about the cohomology and is still open. Jeffrey Manning and I are currently writing up the following result:

Expected Theorem C.2. Suppose that $l > 2$. If $\mathfrak{m}$ is in the support of $\ker(\pi^*)$, then it is exceptional.

'Exceptional' here means that the image of the associated Galois representation is reducible or has order coprime to l; it is equivalent to the projective image not containing $PSL_2(\mathbb{F}_l)$. In the case that Γ is the group of units in an Eichler order of squarefree level, all exceptional maximal ideals are Eisenstein, and so we prove Conjecture C.1. Our method uses patching and careful consideration of local deformation rings at an auxiliary prime q to reduce to the known case of a definite quaternion algebra. It is entirely different from that of Ribet (who uses the congruence subgroup property of $SL_2(\mathbb{Z}[1/p])$, which is not known for quaternion orders) or that of Diamond and Taylor (who use the mod l geometry of X_Γ, leading to constraints on l).

We expect also to be able to use our method to solve the following:

Problem C.3. *Generalise this method to quaternion algebras over totally real fields.*

This would have applications to the anticyclotomic main conjecture of Iwasawa theory for Hilbert modular forms, as explained in [Lon12].

For generalisation to higher rank, it is necessary to take a representation-theoretic perspective. To make the connection, consider the tower of curves

$$\left(X_{\Gamma\cap\Gamma(p^n)}\right)_{n\geq 0}$$

with increasing level structure at p, giving rise to an $\mathbb{F}_l$-vector space

$$V = \varinjlim H^1(X_{\Gamma\cap\Gamma(p^n)}, \mathbb{F}_l)$$

with an action of $GL_2(\mathbb{Q}_p)$. Then Ihara's lemma can be reinterpreted as saying that any one-dimensional subrepresentation of V is Eisenstein. In higher rank, one similarly constructs $\mathbb{F}_l$-representations V of $GL_n(\mathbb{Q}_p)$ from the cohomology of Shimura varieties, and then there is the following important open problem:

Conjecture C.4. *(Conjecture B of [CHT08]) Any nongeneric irreducible subrepresentation of V is Eisenstein.*

Being nongeneric is a technical property of an irreducible representation of $GL_n(\mathbb{Q}_p)$ that, when $n = 2$, is equivalent to being one-dimensional.

We will investigate what patching and the local Langlands correspondence have to say about this conjecture. Fix a non-Eisenstein maximal ideal $\mathfrak{m}$, with associated Galois representation $\rho_{\mathfrak{m}}$, and let $\rho_{\mathfrak{m},p}$ be the restriction of $\rho_{\mathfrak{m}}$ to $G_{\mathbb{Q}_p}$. Let R_p be the universal deformation ring of $\rho_{\mathfrak{m},p}$. Patching, roughly speaking, allows one to construct an R_p-module M_∞ with an action of $GL_n(\mathbb{Q}_p)$. This module has the property that its invariants under any compact open subgroup of $GL_n(\mathbb{Q}_p)$ form a maximal Cohen–Macaulay module over R_p.

Question C.5. *Is M_∞ the interpolation of local Langlands constructed in [HM16]?*

This question is closely related to C.4, although the precise relationship requires some clarification.

Problem C.6. *Use the geometry of R_p to study M_∞, and hence obtain results towards conjecture C.4.*

A related problem is the following. If $\Gamma \cap \Gamma(p)$ is the subgroup of Γ whose elements are trivial mod p, then each irreducible $GL_2(\mathbb{F}_p)$-subrepresentation of $H^1(X_{\Gamma\cap\Gamma(p)}, \mathbb{Q}_l)$ has a canonical lattice coming from the $\mathbb{Z}_l$-cohomology — can one identify this lattice? Of course the same question can be asked for higher rank groups. For Shimura curves in the case $l = p$, this question was asked by Breuil [Bre14] and answered by Emerton, Gee and Savitt [EGS15], using exactly the kind of patching argument that we propose.

D. Finitely-presented mod p representations

Let F be a finite extension of $\mathbb{Q}_p$. Then the mod p local Langlands program — hypothetical if $F \neq \mathbb{Q}_p$ — attempts to relate the smooth irreducible mod p representations of $GL_2(F)$ to the two-dimensional continuous mod p representations of G_F. The Galois representations are easy to classify here, but for $F \neq \mathbb{Q}_p$ the mod p representation theory of $GL_2(F)$ is difficult. Natural constructions lead to infinite length representations, and the answers to even foundational questions are not always known; for instance, it is unknown whether an irreducible smooth representation of $GL_2(F)$ has a central character. I proved the following result about *finitely presented* mod p representations of $GL_2(F)$, answering a question of Emerton.

Theorem D.1. *The kernel of any homomorphism between finitely presented, smooth mod p representations of $GL_2(F)$ is also finitely presented.*

In other words, the finitely presented smooth representations are an abelian subcategory of all smooth representations. The short proof makes novel use of an amalgamated product of completed group rings. It uses that $GL_2(F)$ is an amalgam of two compact open subgroups, and so does not generalise to $GL_n(F)$.

Problem D.2. *Does Theorem D.1 hold for $n > 2$?*

Many irreducible mod p representations of $GL_2(F)$ are known to *not* be finitely presented by [Sch15]. Nonetheless, from the point of view of patching constructions this is a natural category to consider.

In [CEG$^+$16b], patching is used to construct, for any n, a representation M_∞ of $GL_n(F)$ over R_p, the local deformation ring for a fixed mod p representation of G_F. By specialising this at points of the deformation ring, it is possible to construct, for each continuous n-dimensional p-adic representation of G_F, an admissible unitary p-adic Banach space representation of $GL_n(F)$. This is what a possible p-adic local Langlands correspondence would predict. But the construction makes many arbitrary choices, and very little is known about its properties in general.

Problem D.3. *Is M_∞ independent of the choices made in its construction?*

This is known for $GL_2(\mathbb{Q}_p)$ by [CEG$^+$16a], and in no other cases. It seems only to be possible to probe M_∞ directly by considering

$$M_\infty(N) = \mathrm{Hom}_{GL_2(F)}(N, M_\infty^\vee)^\vee$$

for N *finitely presented*; this motivates consideration of the category of such representations. This functor is not exact, and using Theorem D.1 we can study its derived functors.

Problem D.4. *Study the functor $N \mapsto M_\infty(N)$ from $\mathcal{C}$ to $R_p - \mathrm{Mod}$, and its derived functors. Are they independent of the choices?*

This last problem is also interesting in the $l \neq p$ setting, and in this case ought to be accessible if Ihara's lemma (Conjecture C.4) is assumed.

Problem D.5. *Assuming Ihara's lemma, compute the 'derived functors of patching' when $l \neq p$.*

So far I have been able to do this in some cases when $n = 2$. In general I would hope to find relations between the non-genericity of the representations N, the cohomological degrees in which their derived functors are supported, and the dimension of their support on R_p.

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