

RESEARCH STATEMENT — JACK SHOTTON

INTRODUCTION

The Breuil–Mézard conjecture relates the mod p geometry of moduli spaces for representations of p -adic fields to the mod p representation theory of $GL_n(\mathbb{Z}_p)$. It has been influential in the development of automorphy lifting theorems and the p -adic Langlands program. It is natural to ask whether there is an analogue of this conjecture when “mod p ” is replaced by “mod ℓ ” for ℓ a prime *distinct* from p , and indeed the main result of my thesis is:

Theorem A. (vague version) If $\ell \neq p$ and $\ell > 2$, then a natural analogue of the Breuil–Mézard conjecture holds.

This work belongs to the Langlands program, the area of number theory that studies Galois representations and their connections to automorphic representations. If $\overline{\mathbb{Q}}$ is the field of algebraic numbers, then the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of \mathbb{Q} is the group of field automorphisms of $\overline{\mathbb{Q}}$; it is a profinite group. **Galois representations** are continuous finite-dimensional representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with coefficients in the ℓ -adic numbers \mathbb{Q}_ℓ ¹ — that is, continuous homomorphisms

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_n(\mathbb{Q}_\ell).$$

Since the (ℓ -adic) étale cohomology of any algebraic variety over \mathbb{Q} yields a Galois representation, they are of great arithmetic interest. **Automorphic representations** come from the spectral theory of reductive algebraic groups over number fields, and generalise the classical modular forms. It is a guiding principle that there should be a natural bijection:

“algebraic” automorphic representations \longleftrightarrow “geometric” Galois representations.

In many cases, it is now possible to associate Galois representations to automorphic representations. Results in the opposite direction are very incomplete. The most fruitful technique, **automorphy lifting**, was developed by Wiles and Taylor–Wiles in [Wil95] and [TW95] to prove that all semistable² elliptic curves over \mathbb{Q} are modular (and hence deduce Fermat’s last theorem). The generalisation of Wiles’ techniques to n -dimensional representations was instrumental in the proof of the Sato–Tate conjecture, which predicts the distribution (as ℓ varies) of the number of mod ℓ points on an elliptic curve. For these generalisations, it is crucial to use Kisin’s point of view on automorphy lifting theorems (see [Kis09b]). This brings the moduli spaces of *local* Galois representations into the foreground; by studying their arithmetic–geometric properties, it is possible to prove interesting results about congruence properties of automorphic forms. My work is on the interplay between these local deformation rings and the representation theory of p -adic matrix groups.

¹or in a finite extension of \mathbb{Q}_ℓ

²This condition was subsequently removed by Breuil, Conrad, Diamond and Taylor in [BCDT01].

THE BREUIL–MÉZARD CONJECTURE WHEN $\ell \neq p$.

Suppose that ℓ and p are distinct primes. Let E/\mathbb{Q}_ℓ be a finite extension, \mathcal{O} be its ring of integers with uniformiser λ , and \mathbb{F} be the residue field of \mathcal{O} ; we will assume in the following that E is ‘sufficiently large’. Let F/\mathbb{Q}_p be a finite extension with ring of integers \mathcal{O}_F and residue field k_F . Fix a continuous representation

$$\bar{\rho} : \text{Gal}(\bar{F}/F) \longrightarrow GL_n(\mathbb{F}).$$

Then there is a universal lifting ring $R^\square(\bar{\rho})$ together with a universal lift

$$\rho^\square : \text{Gal}(\bar{F}/F) \longrightarrow GL_n(R^\square(\bar{\rho})).$$

The fibres of the structure morphism

$$\text{Spec } R^\square(\bar{\rho}) \longrightarrow \text{Spec } \mathcal{O}$$

are equidimensional of dimension n^2 .

We can partially describe the irreducible components of $\text{Spec } R^\square(\bar{\rho}) \otimes E$. Let I_F be the inertia subgroup of $\text{Gal}(\bar{F}/F)$. If \mathcal{C} is an irreducible component of $\text{Spec } R^\square(\bar{\rho}) \otimes E$ then the isomorphism class of $\rho^\square|_{I_F}$ is constant on an open subset of \mathcal{C} ; call this the **inertial type** of the component.

Let Cpts be the free abelian group on the irreducible components of $\text{Spec } R^\square(\bar{\rho}) \otimes E$ and let Groth be the Grothendieck group of finite-length E -representations of $GL_n(\mathcal{O}_F)$. In [Sho14], I define a homomorphism

$$\text{cyc}_E : \text{Groth}_E \longrightarrow \text{Cpts}_E.$$

In general this is a little complicated; one case in which it is straightforward is when τ is a semisimple representation of I_F and $\sigma(\tau)$ is the representation of $GL_n(\mathcal{O}_F)$ associated to it by the ‘inertial local Langlands correspondence’. In that case,

$$\text{cyc}(\sigma(\tau)) = \sum \{\text{irreducible components whose inertial type has semisimplification } \tau.\}$$

Now let $\text{Cpts}_\mathbb{F}$ be the free abelian group on the irreducible components of $\text{Spec } R^\square(\bar{\rho}) \otimes_\mathcal{O} \mathbb{F}$, and let $\text{Groth}_\mathbb{F}$ be the Grothendieck group of finite-length \mathbb{F} -representations of $GL_n(\mathcal{O}_F)$. Then the $\ell \neq p$ analogue of the Breuil–Mézard conjecture is:

Theorem A. Suppose that $\ell > 2$. Then there exists a (unique) map

$$\text{cyc}_\mathbb{F} : \text{Groth}_\mathbb{F} \longrightarrow \text{Cpts}_\mathbb{F}$$

compatible with cyc_E via the natural reduction maps $\text{Groth}_E \rightarrow \text{Groth}_\mathbb{F}$ and $\text{Cpts}_E \rightarrow \text{Cpts}_\mathbb{F}$.

Proof. In [Sho13] I showed that this is true when $n = 2$ using explicit local methods, by giving equations for the irreducible components of $\text{Spec } R^\square(\bar{\rho})$. In [Sho14], I use global methods similar to those of Emerton and Gee [EG14]. The idea is this: First, use potential automorphy methods to realise $\bar{\rho}$ as a local component of some

$$\bar{\tau} : \text{Gal}(\bar{K}/K) \rightarrow GL_n(\mathbb{F})$$

for a global field K , such that $\bar{\tau}$ comes from an automorphic representation. Then use the Taylor–Wiles–Kisin patching method to obtain an exact functor from the category of finitely-generated smooth $\mathcal{O}[GL_n(\mathcal{O}_F)]$ -modules to the category of finitely-generated $R^\square(\bar{\rho})$ -modules that is ‘compatible with reduction mod ℓ ’. The theorem follows by looking at the support of these modules.

Example. Let me illustrate theorem A in the case $n = 2$, $\bar{\rho}$ unramified and $\#k_F \equiv 1 \pmod{\ell}$. Construct three E -representations $\mathbb{1}$, St and PS of $GL_2(k_F)$ as follows: $\mathbb{1}$ is the trivial representation, St is the Steinberg representation (i.e. the quotient of the space of E -valued functions on $\mathbb{P}_{k_F}^1$ by the constant functions), and PS is the induction from a Borel subgroup of $GL_2(k_F)$ of a non-trivial character that is congruent to 1 mod ℓ . Regard these as representations of $GL_2(\mathcal{O}_F)$. Then the following congruence holds (interpreted as an equality in $\text{Groth}_{\mathbb{F}}$):

$$\mathbb{1} \oplus \text{St} \equiv \text{PS} \pmod{\lambda}.$$

Let \mathcal{C}_{nr} be the component of $\text{Spec } R^{\square}(\bar{\rho}) \otimes E$ on which ρ^{\square} is unramified, \mathcal{C}_{unip} be the component on which $\rho^{\square}(I_F)$ is unipotent but (generically) ramified, and \mathcal{C}_{ram} be the component on which $\rho^{\square}|_{I_F}$ is a sum of distinct characters. Then cyc maps:

$$\begin{aligned} \mathbb{1} &\longmapsto \mathcal{C}_{nr} \\ \text{St} &\longmapsto \mathcal{C}_{nr} + \mathcal{C}_{unip} \\ \text{PS} &\longmapsto \mathcal{C}_{ram}. \end{aligned}$$

Theorem A then predicts that we have a congruence (i.e. equality of reductions in $\text{Cpts}_{\mathbb{F}}$):

$$\mathcal{C}_{unip} + 2\mathcal{C}_{nr} \equiv \mathcal{C}_{ram} \pmod{\lambda}$$

and in [Sho13] I show that this is indeed true.

Motivation. The original Breuil–Mézard conjecture was posed for $\ell = p$, $n = 2$ and $F = \mathbb{Q}_p$. It was (mostly) proved by Kisin in [Kis09a] as part of his proof of most cases of the Fontaine–Mazur conjecture for GL_2/\mathbb{Q} . The formulation for general n and F , using cycles on $R^{\square}(\bar{\rho})$, is due to Emerton and Gee [EG14]. The key differences from the $\ell \neq p$ case are:

- on the left hand (“automorphic”) side, one must use all locally algebraic representations of $GL_n(\mathcal{O}_F)$ rather than just the smooth representations (those with open kernel);
- on the right hand (“Galois”) side, one must use cycles of positive codimension on $R^{\square}(\bar{\rho})$ — these are labelled by p -adic Hodge theory data and were constructed by Kisin in [Kis08].

Another important difference when $\ell = p$ is that everything is very much conjectural for $n > 2$!

The other motivation for this work comes from Taylor’s Ihara avoidance method (see [Tay08]). Without wishing to go into too much detail, the basic problem is that the Taylor–Wiles–Kisin method seems to only be able to deduce automorphy for global representations that lie on the same component of a local deformation ring as a representation that is known to be automorphic; to move automorphy between different components is to identify which weights and levels a global mod ℓ representation is automorphic in. In [CHT08] it was only possible to solve this problem using a conjectural statement about mod ℓ automorphic forms known as “Ihara’s lemma”. Taylor’s argument showed that automorphy could be propagated between different components of the local deformation ring ‘through the special fibre’. This, in effect, requires a close study of the reduction map $\text{Cpts}_E \rightarrow \text{Cpts}_{\mathbb{F}}$ in a particular special case.

Further work. There are several natural follow on questions that I intend to work on. The first is:

Problem. Give a purely local proof of theorem A.

Hopefully such a proof would also apply when $\ell = 2$; it would also give more insight into why theorem A is true. A possible strategy for giving a local proof is the following. It is possible to reduce to the case where $\bar{\rho}$ is tamely ramified, and in this case $R^\square(\bar{\rho})$ can be obtained as the completion of the ring of functions at a point in the special fibre of an affine scheme $\mathcal{M}(n, \#k_F)$ of finite type over $\text{Spec } \mathcal{O}$. Explicitly, $\mathcal{M}(n, \#k_F)$ is the space of pairs of invertible matrices (Σ, Φ) such that

$$\Phi \Sigma \Phi^{-1} = \Sigma^{\#k_F}.$$

Assuming that $\bar{\rho}$ corresponds to a ‘sufficiently generic’ point of the special fibre simplifies $R^\square(\bar{\rho})$ considerably; I then hope to be able to deduce the general case by variation on $\mathcal{M}(n, \#k_F)$. As evidence that the first part of this strategy, at least, could work, I have found a local proof in the case where $\#k_F \equiv 1 \pmod{\ell}$, $\bar{\rho}$ is unramified, and $\bar{\rho}(\text{Frob}_F)$ has pairwise distinct eigenvalues.

There are also several natural geometric questions about the schemes $R^\square(\bar{\rho})$ that are unanswered. For example, we have:

Theorem B. Suppose that $\ell > 2$ and $n = 2$.

- (1) [Pil] The irreducible components of $\text{Spec } R^\square(\bar{\rho}) \otimes E$ are smooth.
- (2) [Sho13] Fix a semisimple representation τ of I_F . If $R^\square(\bar{\rho}, \tau)$ is the maximal λ -torsion-free quotient of $R^\square(\bar{\rho})$ on which $\bar{\rho}|_{I_F}^{ss} \cong \tau$, then $R^\square(\bar{\rho}, \tau)$ is Cohen–Macaulay.

Knowing that a local deformation ring is Cohen–Macaulay can have global consequences, as explained in [Sno11]. It is known by work of Helm that $R^\square(\bar{\rho})$ is a complete intersection, and so Cohen–Macaulay; however, in general the quotients in theorem B part 2 are not Gorenstein, so not complete intersections.

Question. Is theorem B true for $n > 2$?

The proof of part 2 of theorem B for $n = 2$ uses the fact that determinantal rings are Cohen–Macaulay; it is my hope that something similar will work for higher n . Part 1 should be quite accessible; the key point is to find ‘enough’ equations that hold on each component of $\mathcal{M}(n, \#k_F)$.

LOCAL LANGLANDS IN FAMILIES

The local Langlands correspondence associates to each continuous representation $\rho : \text{Gal}(\bar{F}/F) \rightarrow GL_n(E)$ an irreducible admissible E -representation $\pi(\rho)$ of $GL_n(E)$. When one has a family of Galois representations living over a deformation ring, it is natural to ask whether the local Langlands correspondence can be interpolated. More precisely, suppose that A is a complete noetherian local \mathcal{O} -algebra with maximal ideal \mathfrak{m} and residue field \mathbb{F} , and that

$$\rho : \text{Gal}(\bar{F}/F) \longrightarrow GL_n(A)$$

is a continuous representation. In [EH11], Emerton and Helm conjecture:

Conjecture C. Then there is a smooth admissible representation $\pi(\rho)$ of $GL_n(F)$ over A such that $\pi(\rho)$ is A -torsion-free and:

- $\pi(\rho)$ interpolates the dual of (a certain modification of) the local Langlands correspondence;
- $\pi(\rho) \otimes_A \mathbb{F}$ is *essentially absolutely irreducible and generic*.

“Essentially absolutely irreducible and generic” is an important technical condition, that I won’t define here. The motivation for the conjecture is global: it should appear in tensor product factorisations for the completed cohomology of towers of Shimura varieties, where A is taken to be a suitably completed global Hecke algebra. For the tower of modular curves, this was shown (under technical hypotheses) by Emerton in [Eme11].

Problem. Use Taylor–Wiles–Kisin patching to prove conjecture C.

My proof of theorem A, as well as the candidate for the p -adic Langlands correspondence constructed in [CEG⁺13], suggest that it may be possible to prove conjecture C in this way. Carrying this out is an ongoing project of mine joint with Emerton, Gee and Helm.

It suffices to prove conjecture C in the case where $A = R^\square(\bar{\rho})$ and ρ is the universal lift. The Taylor–Wiles–Kisin machine can be used to produce an $R^\square(\bar{\rho})[GL_n(F)]$ -module M_∞ satisfying all of the conditions of conjecture C except perhaps for the last. The hope is that a combination of the geometry of $R^\square(\bar{\rho})$, commutative algebra of M_∞ , and representation theory of $GL_n(F)$ can be used to show that the last condition also holds. In fact, using work of Helm, it would be possible to deduce conjecture C if we knew the following, weaker, property of M_∞ :

$$\mathrm{End}_{R^\square(\bar{\rho})[GL_n(F)]}(M_\infty) = R^\square(\bar{\rho}).$$

It is likely that any global proof of conjecture C would also shed light on:

Problem. Show that the correspondence of conjecture C occurs in the cohomology of Shimura varieties.

This is a structural result on the \mathbb{F}_ℓ -cohomology of Shimura varieties related to Ihara’s lemma, but the precise connection requires clarification and this is another goal of our project.

DEFORMATION RINGS FOR NON-REGULAR WEIGHTS

Suppose that $F = \mathbb{Q}_p$ and E is a finite extension of \mathbb{Q}_p . Suppose that $\bar{\rho} : \mathrm{Gal}(\bar{F}/F) \rightarrow GL_n(\mathbb{F})$ is a continuous representation. If $\underline{k} = (k_1 \geq \dots \geq k_n)$ is a tuple of integers (a **weight**), then (by [Kis08]) there is a p -torsion-free quotient

$$R^\square(\bar{\rho}) \twoheadrightarrow R_{cr, \underline{k}}^\square(\bar{\rho})$$

parameterizing (in a suitable sense) lifts of ρ that are crystalline with Hodge–Tate weights k_1, \dots, k_n . If k_1, \dots, k_n are distinct then we say that the weight is regular; in this case, $R_{cr, \underline{k}}^\square(\bar{\rho})$ has relative dimension $n^2 + \frac{n(n-1)}{2}$ over \mathcal{O} ; if the weight is not regular, then the dimension of $R_{cr, \underline{k}}^\square(\bar{\rho})$ is smaller.

I intend to work on:

Problem. Suppose that \underline{k} is non-regular. Describe the intersection of $R_{cr, \underline{k}}^\square \otimes \mathbb{F}$ and $R_{cr, \underline{k}'}^\square \otimes \mathbb{F}$ (where we may take \underline{k}' to be regular and ‘small’).

This is in the spirit of the Breuil–Mézard conjecture, which compares the special fibres of $R_{cr, \underline{k}}^\square(\bar{\rho}) \otimes \mathbb{F}$ as \underline{k} varies over the regular weights.

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