YGGTX

Measure Equivalence, Superrigidity, and Weak Pinsker Entropy

Robin Tucker-Drob

Based on joint work with Lewis Bowen

DEF Two groups Γ & Λ are called <u>measure equivalent</u> (ME) if they admit an <u>ME-coupling</u>, i.e., a standard measure space (Ω, μ) equipped with measure preserving actions $\Gamma \curvearrowright (\Omega, \mu)$ & $\Lambda \curvearrowright (\Omega, \mu)$ such that: (1) the actions commute, i.e., $\gamma \cdot (\lambda \cdot \omega) = \lambda \cdot (\gamma \cdot \omega)$ for $\gamma \in \Gamma$, $\lambda \in \Lambda$
& (2) each action admits a finite measure <u>fundamental domain</u>, i.e., there are finite measure sets F_Γ & F_Λ such that $(\gamma \cdot F_\Gamma)_{\gamma \in \Gamma}$ partitions Ω and likewise $(\lambda \cdot F_\Lambda)_{\lambda \in \Lambda}$ partitions Ω.

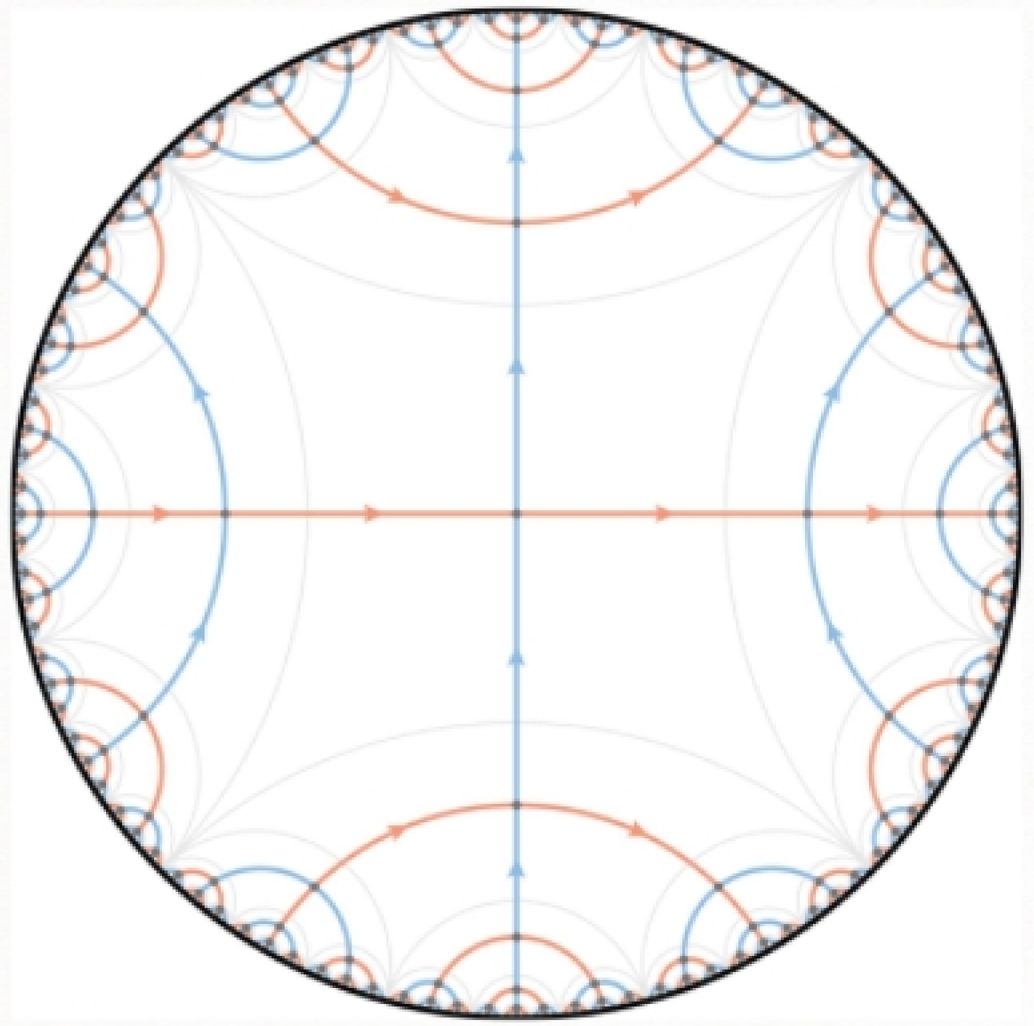
The <u>index</u> of the coupling is the number $[\Gamma : \Lambda]_\Omega := \frac{\mu(F_\Lambda)}{\mu(F_\Gamma)}$

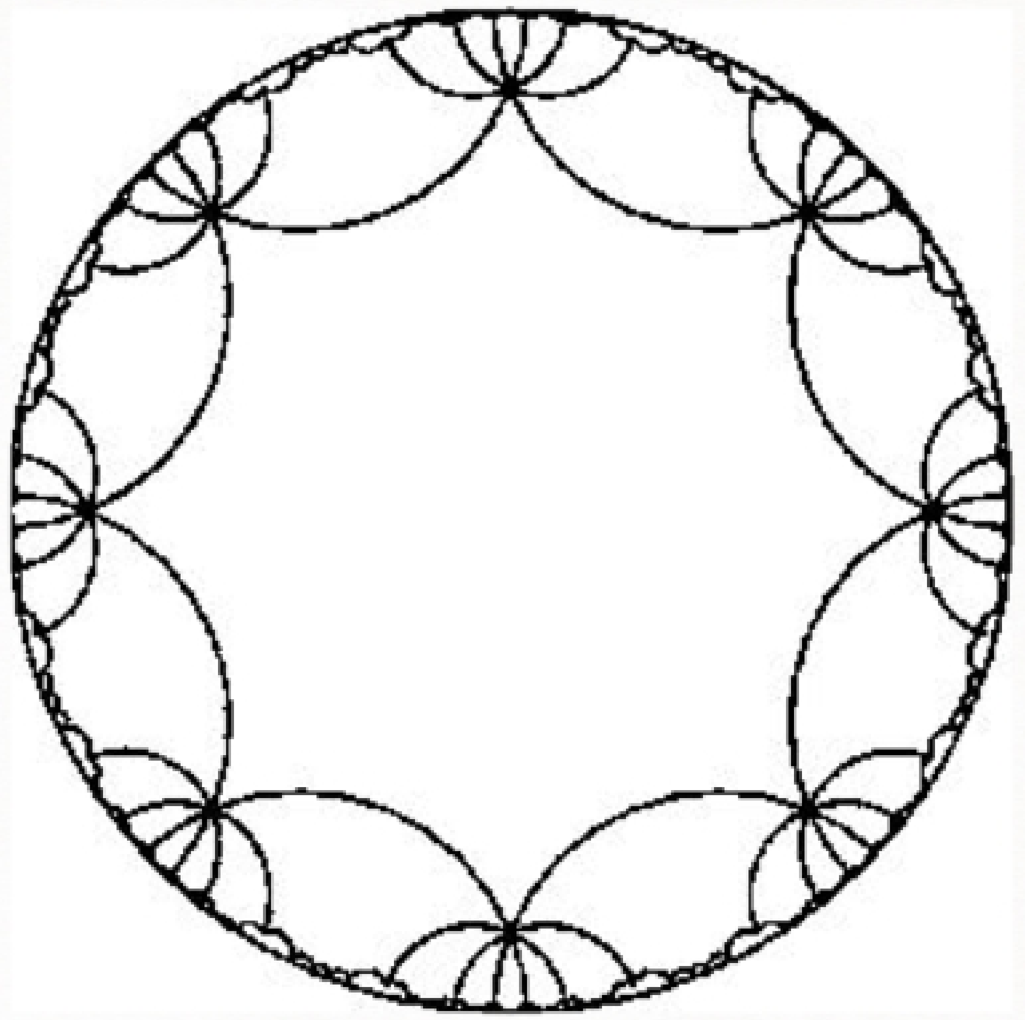
Ex Suppose Λ is a finite index subgroup of Γ. Take $\Omega = \Gamma$ and μ = counting measure. Let Γ & Λ act by left & right translation respectively. Take $F_\Gamma = \{e\}$ & F_Λ = any set of left coset representatives for Λ in Γ
Then $[\Gamma : \Lambda]_\Omega = [\Gamma : \Lambda]$

DEF A <u>lattice</u> in a locally compact second countable group G is a discrete subgroup Γ of finite covolume.

Thus, if Γ & Λ are both lattices in the same locally compact group G then Γ & Λ are ME since we can take $(\Omega, \mu) = (G, \mu_G)$ with Γ & Λ acting by left and right translation. Then $[\Gamma : \Lambda]_G = \dfrac{\mathrm{covol}_G(\Lambda)}{\mathrm{covol}_G(\Gamma)}$

Ex The free group $F_2 = \mathbb{Z} * \mathbb{Z}$ & the surface group $\Sigma_2 = \pi_1(\quad)$ are both lattices in $PSL_2(\mathbb{R})$, hence are ME.





DEF Let $\Gamma \curvearrowright (X, \mu_X)$ & $\Lambda \curvearrowright (Y, \mu_Y)$ be measure preserving actions of Γ & Λ on probability spaces. The actions are said to be <u>stably orbit equivalent</u> if there exist positive measure sets $A \subseteq X$ and $B \subseteq Y$ and a measure space isomorphism $\varphi : (A, \mu_A) \longrightarrow (B, \mu_B)$ such that

$\varphi(\Gamma \cdot x \cap A) = \Lambda \cdot \varphi(x) \cap B$ for a.e. $x \in A$.

In the case that $A = X$ and $B = Y$ we say the actions are <u>orbit equivalent</u>

PROP (Furman, Gromov)

Γ & Λ are ME iff they admit free probability measure preserving actions which are stably orbit equivalent.

THM (Hjorth): Γ is ME to a free group iff Γ admits a free probability measure preserving action $\Gamma \curvearrowright (X,\mu)$ which is treeable, i.e., there is a measurable assigment of trees to the Γ-orbits.

Cheating Def For Γ as above, the cost of Γ is defined by

$$\mathrm{cost}(\Gamma) = \frac{1}{2}\int_X \mathrm{degree}_\tau(x)\, d\mu$$

Gaboriau: this does not depend on the choice of treeable action, or on the assignment τ !!

ME Class	Cost Description	Group Theoretic Description
class of $F_0 = \{e\}$	treeable $\mathrm{cost}(\Gamma) < 1$	finite groups
class of $F_1 = \mathbb{Z}$	treeable $\mathrm{cost}(\Gamma) = 1$	infinite amenable groups
class of F_2	treeable $1 < \mathrm{cost}(\Gamma) < \infty$	? ? ? ? ?
class of F_∞	treeable $\mathrm{cost}(\Gamma) = \infty$	? ? ? ? ?

OPEN PROBLEM!

Invariants of ME

- finiteness.
- amenability. Moreover (Ornstein-Weiss) all infinite amenable groups are ME.
- property (T)
- the Haagerup Approximation Property
- Gaboriau: Let (Ω, μ) be an ME-coupling of Γ & Λ. Then

$$\beta_n^{(2)}(\Lambda) = [\Gamma : \Lambda]_\Omega \, \beta_n^{(2)}(\Gamma) \quad \& \quad \mathrm{Cost}(\Lambda) - 1 = [\Gamma : \Lambda]_\Omega (\mathrm{Cost}(\Gamma) - 1)$$

- Cowling-Haagerup constant, weak amenability
- Exactness (Ozawa) bi-exactness (Sako).
- $H_b^2(\Gamma, \ell^2(\Gamma)) \neq \{0\}$ (Monod-Shalom).
- proper proximality (Ishan-Peterson-Ruth) (introduced by Boutonnet-Ioana-Peterson)

COCYCLES

DEF Let $\Gamma \curvearrowright (X,\mu)$ be a pmp action and let L be a countable group.
An <u>L-valued cocycle</u> of the action is a measurable map
$\alpha : \Gamma \times X \longrightarrow L$ satisfying the cocycle identity: $\alpha(\gamma\delta, x) = \alpha(\gamma, \delta\cdot x)\alpha(\delta, x)$
Two L-valued cocycles α & β are called <u>cohomologous</u> if there exists a measurable $f : X \longrightarrow L$ such that $f(\gamma\cdot x)\alpha(\gamma, x)f(x)^{-1} = \beta(\gamma, x)$.

Ex 1) If $\pi : \Gamma \to L$ is a group homomorphism then $\alpha_\pi(\gamma, x) := \pi(\gamma)$ is a cocycle.
2) Suppose $\varphi : (X,\mu) \longrightarrow (Y,\nu)$ is an orbit equivalence from the Γ-action to a free pmp action $\Lambda \curvearrowright (Y,\nu)$. The associated <u>Zimmer cocycle</u> $\alpha_\varphi : \Gamma \times X \longrightarrow \Lambda$ is given by: $\alpha_\varphi(\gamma, x)$ is the unique element in Λ satisfying $\alpha_\varphi(\gamma, x)\cdot\varphi(x) = \varphi(\gamma\cdot x)$

DEF Let (K,μ_K) be a probability space. The <u>Bernoulli shift</u> of Γ with base (K,μ_K) is the free pmp action $\Gamma \curvearrowright (K^\Gamma, \mu_K^\Gamma)$ given by

$$(\gamma \cdot z)(\delta) = z(\gamma^{-1}\delta) \quad \text{for } \gamma,\delta\in\Gamma,\ z\in K^\Gamma$$

Popa's Cocycle Superrigidity Theorem

Let Γ be either (1) A group with property (T)

or (2) A product $\Gamma = H \times A$ with H nonamenable & A infinite.

Then: For every countable group L:

Given any pmp action $\Gamma \curvearrowright (X,\mu)$, every L-valued cocycle of the product action $\Gamma \curvearrowright (K^\Gamma, \mu^\Gamma) \times (X,\mu)$ is cohomologous to a cocycle that does not depend on the Bernoulli coordinate.

DEF Γ is called <u>Bernoulli superrigid</u> if it satisfies the conclusion of Popa's Theorem.

Bernoulli superrigid groups, examples:

Prop.
- property (T) groups and product groups $H \times A$ with H nonamenable & A infinite
- If N is normal in Γ and N is Bernoulli superrigid then Γ is as well.

- Peterson-Sinclair: If $L\Gamma$ has property gamma then Γ is Bernoulli superrigid
- Ioana-TD: Nonamenable Inner amenable groups are Bernoulli superrigid.

$\boxed{\text{THM (Bowen-TD)} \mid \text{Bernoulli Superrigidity is an invariant of ME}}$

COROLLARY (of our more general theorem)

Let Γ be a nonamenable group which is a lattice in a product of two noncompact locally compact groups. Then Γ is Bernoulli superrigid

- Peterson-Sinclair: Γ is Bern. superrigid $\Longrightarrow \beta_1^{(2)}(\Gamma) = 0$ & Γ is nonamenable

Open Problem: Does the converse hold?

DEF Suppose that the pmp action $\Gamma \curvearrowright (X,\mu)$ is isomorphic to a product $\Gamma \curvearrowright (K^\Gamma, \mu_K^\Gamma) \times (Y,\nu)$ of a Bernoulli action with another action. In this case we call $\Gamma \curvearrowright (K^\Gamma, \mu_K^\Gamma)$ a <u>direct Bernoulli factor</u> of $\Gamma \curvearrowright (X,\mu)$.

The <u>weak Pinsker entropy</u> of $\Gamma \curvearrowright (X,\mu)$ is defined by:

$$h^{wP}(\Gamma \curvearrowright (X,\mu)) := \sup \left\{ H(K,\mu_K) \;\middle|\; \begin{array}{l} \Gamma \curvearrowright (K^\Gamma,\mu_K^\Gamma) \text{ is a direct Bernoulli} \\ \text{factor of } \Gamma \curvearrowright (X,\mu) \end{array} \right\}$$

where $H(K,\mu_K) := -\sum_{k\in K} \mu_K(\{k\}) \log(\mu_K(\{k\}))$ is Shannon entropy

$\left[H(K,\mu_K) := +\infty \text{ if } (K,\mu_K) \text{ is not purely atomic.} \right]$

THM (Bowen–TD) Let Γ be a Bernoulli superrigid group and let (Ω, μ) be an ME-coupling of Γ with a group Λ. Then

$$h^{wP}(\Lambda \curvearrowright (\Omega/\Gamma, \mu_{\Omega/\Gamma})) = [\Gamma : \Lambda]_\Omega \, h^{wP}(\Gamma \curvearrowright (\Omega/\Lambda, \mu_{\Omega/\Lambda}))$$

As a consequence: for actions of Γ, weak Pinsker entropy is an invariant of orbit equivalence.

Thank You!