A class of increasing homeomorphism groups naturally isomorphic to diagram groups

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Geometrically Fast Sets

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We say B is **geometrically fast** if there exists a marking for B such that its feet are pairwise disjoint. (C. Bleak et. al [1])

Two bumps with non-trivial overlap

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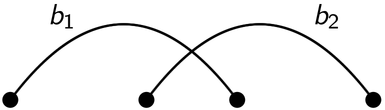

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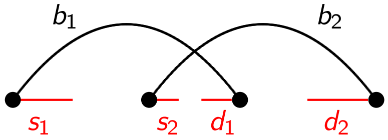

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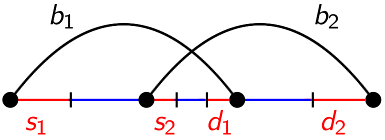

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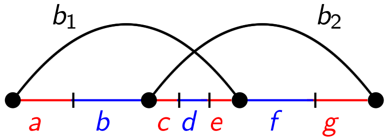

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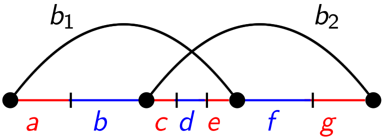

Using this partition, we have that

$$(a)b_1 = abcd \qquad (c)b_2 = cdef$$
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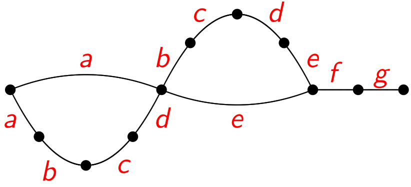
β_1



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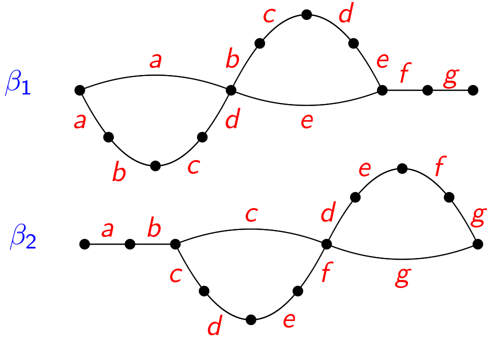

Bumpy diagram groups

We can see that $\langle \beta_1, \beta_2 \rangle$ is a subgroup of the diagram group $D(\mathcal{P}_{\{b_1,b_2\}}, abcdefg)$ and the assignment $b_1 \mapsto \beta_1, b_2 \mapsto \beta_2$ extends to a map $\delta : \langle b_1, b_2 \rangle \to D(\mathcal{P}_{\{b_1,b_2\}}, abcdefg)$.

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$$\begin{aligned}\mathcal{P}_B = \langle a_1, \ldots, a_{4n-1} \mid (a_k, a_k a_{k+1} \ldots a_{l-1}),& \\ (a_{k+1} \ldots a_{l-1} a_l, a_l) &\text{ for each } b_i \in B \rangle\end{aligned}$$

where a_k and a_l is the source and destination of each b_i respectively.

References

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