yGGTX Parallel Sessions -

Limit groups

Jonathan Fruchter

University of Oxford

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Overview

1. What are *limit groups* - some history and examples
2. Limit groups are limits of free groups (and some logic)
3. Hierarchies
4. Properties of limit groups
5. Generalizations

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Residual properties

Limit groups have been studied since the 1960's (Baumslag, Lyndon and others) under the name *finitely generated fully residually free groups*.

Recall that a group G is *residually free* if for every $1 \neq g \in G$ there is a homomorphism $f : G \to F$ such that $f(g) \neq 1$.

Definition

A group G is called *fully* residually free (or ω-residually free) if for every finite subset $A \subset G$ there is a homomorphism $f : G \to F$ that is injective on A.

Remark

A finitely generated subgroup of a limit group is also a limit group.

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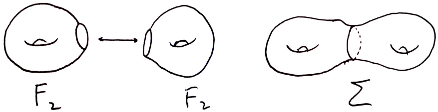

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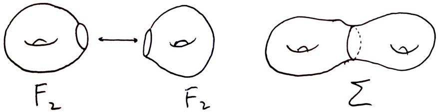
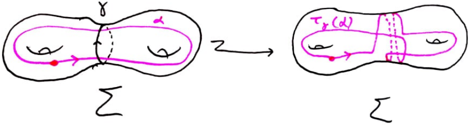


Let $r : \pi_1\Sigma \to F_2$ be the retraction which maps the right half surface to the left one.
Let $\tau_\gamma : \pi_1\Sigma \to \pi_1\Sigma$ be the automorphism of $\pi_1\Sigma$ which restricts to the identity on the left half surface, and to conjugation by the loop γ on the right half surface:



Examples - continued

More formally, $\tau_\gamma(\alpha) = \begin{cases} \alpha & \alpha \text{ in left half surface} \\ \gamma\alpha\gamma^{-1} & \alpha \text{ in right half surface} \end{cases}$
and for $a_1 b_1 \cdots a_n b_n \in \pi_1(\Sigma)$ where a_i and b_i lie in $\pi_1(\text{left})$ and $\pi_1(\text{right})$ respectively,

$$\tau_\gamma(a_1 b_1 \cdots a_n b_n) = a_1\ \gamma\ b_1\ \gamma^{-1} \cdots a_n\ \gamma\ b_n\ \gamma^{-1}.$$

The map $r \circ (\tau_\gamma)^k : \pi_1\Sigma \to F_2$ sends $a_1 b_1 \cdots a_n b_n \in \pi_1(\Sigma)$ to

$$r(a_1)\ [x,y]^k\ r(b_1)\ [x,y]^{-k} \cdots r(a_n)\ [x,y]^k\ r(b_n)\ [x,y]^{-k}$$

which is nontrivial for large enough k ("boundary games").

Recommendation

Dense embeddings of surface groups (Emmanuel Breuillard, Tsachik Gelander, Juan Souto, Peter Storm, '06).

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4. Extensions of centralizers:

Definition

Let G be a limit group and let $g \in G$. The *extension of the centralizer* $C(g)$ by a free abelian group A is the group

$$G *_{C(g)} (C(g) \times A).$$

As before, define a map

$$f : G *_{C(g)} (C(g) \times A) \to G$$

which restricts to the identity on G and which maps A to large powers of g. This shows that $G *_{C(g)} (C(g) \times A)$ is fully residually G (hence fully residually free).

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Marked groups

Definition

A *marked group* is a pair (G, S) such that G is a group and S is a finite generating set of G. Define $\mathcal{G}_n$ to be the set of marked groups (G, S) such that $|S| = n$.

The space $\mathcal{G}_n$ is a metric space:

$$d((G,S),(G',S')) = e^{-N}$$

where N is the maximal integer such that radius N balls around 1 in $X(G,S)$ and $X(G',S')$ are the same.

Lemma

G is a limit group $\iff$ $G = \lim_{i\to\infty}(G_i, S_i)$ in $\mathcal{G}_n$ and each G_i is a free group.

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Some connections with logic

Recall that the *first order theory* of a group consists of the sentences of the form

$$\forall x_1, \ldots, x_{n_x} \exists y_1, \ldots, y_{n_y} \forall z_1, \ldots, z_{n_z} \cdots \bigvee_{i=1}^{k} \bigwedge_{j=1}^{m} w_{i,j}\left(x_1, x_2, \ldots\right) \overset{\neq}{=} 1$$

which are true in G.

For example,

- if G is nontrivial, the sentence $\exists g\ g \neq 1$ is in $\mathrm{Th}(G)$
- if A is abelian, the sentence $\forall x \forall y\ [x, y] = 1$ is in $\mathrm{Th}(A)$
- if H is torsion-free, the family of sentences $\Phi_n = \forall x\ \left(x \neq 1 \longrightarrow x^n \neq 1\right)$ is in $\mathrm{Th}(H)$

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The universal theory

Theorem

A finitely generated group G is a (non-abelian) limit group $\iff$ G has the same universal (only $\forall$ quantifiers) as a (non-abelian) free group.

sketch-of-Proof.

$\implies$ we will show: if Φ is a universal sentence, the set $\{(G, S)|\ G \models \Phi\}$ is closed in $\mathcal{G}_n$. Equivalently, the set $X = \{(G, S)|\ G \models \neg\Phi\}$ is open. For simplicity, assume $\neg\Phi = \exists x_1 \cdots \exists x_n\ w(x_1, \dots, x_n) = 1$ and $G \models \neg\Phi$. So there are $g_1, \dots, g_n \in G$ s.t $w(g_1, \dots, g_n) = 1$. Let $R > |w(g_1, \dots, g_n)|$, so the ball of radius e^{-R} in $\mathcal{G}_n$ is in X.

$\impliedby$ let $R > 0$, the ball of radius R in G can be encoded by a collection Φ of equations and inequations. There is a free group F satisfying $\exists x_1 \cdots \exists x_n \Phi$ which implies that G and F are at least e^{-R} close.

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Structure of limit groups

These connections with logic drove Sela and independently Kharlampovich-Miyasnikov to study limit groups further, and they played an important role in their positive answer of the following question:

Question (Tarski's question)

Do the first order theories of all non-abelian free groups coincide?

Actions of limit groups on real trees (a topic for another day) play a major role in their proofs. Another key ingredient in their proof is the hierarchical structure of limit groups.

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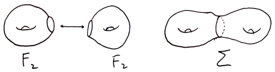
Iterated doubles

Recall the example of a surface group from earlier:



$\pi_1\Sigma$ can be obtained by doubling a free group: $\pi_1\Sigma = F_2 *_{\langle [x,y] \rangle} F_2$.
All limit groups can be obtained by iterating a similar construction:

Definition

A *generalized double* over a limit group G is a group $H = A *_C B$ (or $A*_C$) such that:

1. A, B are finitely generated.
2. C is a non-trivial and maximal abelian in both A and B.
3. $\exists$epimorphism $f : H \to G$ such that $f|_A$ and $f|_B$ are injective.

Theorem

G is a limit group $\iff$ G can be obtained by repeatedly taking generalized doubles (and free products), starting with free groups.

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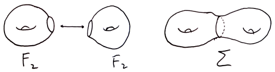

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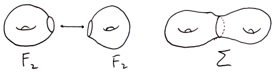

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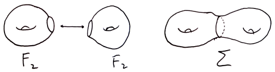

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Iterated centralizer extensions

We already mentioned that if G is a limit group, then so is the centralizer extension $G *_{C(g)} (C(g) \times A)$.

Definition

A group G is an $\mathcal{ICE}$-group (iterated extension of centralizers) if it can be obtained from a free group by a finite sequence of extensions of centralizers.

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The next characterization of limit groups is slightly more complicated, but includes a complete classification of all f.g groups G s.t $\mathrm{Th}(G) = \mathrm{Th}(F)$.

Definition

An *$\omega-$residually free tower* is a space $X = X_n$, constructed floor by floor:

1. The first floor X_0 is a wedge of graphs, (multi-dimensional) tori, and closed hyperbolic surfaces ($\chi < -1$).
2. X_{m+1} is obtained from X_m by attaching a floor of one of the following kinds:
   1. surface: a hyperbolic compact surface Σ with boundary, attached to X_m by its boundary, such that there is a retraction $r : X_{m+1} \to X_m$ with $r_*(\pi_1\Sigma)$ non-abelian.
   2. torus: a torus T^k attached along one of its coordinates (i.e $\{1\} \times \cdots \times S^1 \times \cdots \times \{1\}$), such that the attaching curve generates a maximal abelian subgroup of X_m.

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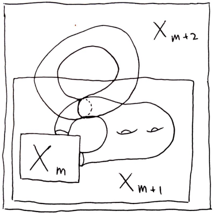

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(Sela) G is a limit group $\iff$ it is a finitely generated subgroup of the fundamental group of an $\omega-$residually free tower.

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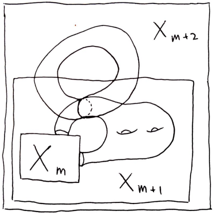
(Sela) Let G be a f.g group. $Th(G) = Th(F) \iff$ G is the fundamental group of an $\omega-$residually free tower whose construction involves no tori.

$\omega-$residually free towers



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Properties of limit groups

1. Limit groups are torsion-free ($\forall x \ (x \neq 1 \to x^n \neq 1)$).
2. Limit groups are commutative transitive ($\forall x \forall y \forall z \ ([x, y] = 1 \wedge [y, z] = 1 \to [x, z] = 1)$).
3. Any two elements of a limit group generate one of the following groups: $\{1\}, \mathbb{Z}, \mathbb{Z}^2, F_2$.
4. Limit groups are hyperbolic relative to free abelian groups (Dahmani).
5. Limit groups are virtually special (Wise).
6. Finitely generated subgroups of limit groups are separable (closed in the profinite topology). Limit groups also virtually retract onto their finitely generated subgroups (Wilton).

Recommendation

Google "Marshall Hall's Theorem"!

7. Every one-ended limit group contains a surface subgroup (Wilton).

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Recommendation

Google "Marshall Hall's Theorem"!

7. Every one-ended limit group contains a surface subgroup (Wilton).

Properties of limit groups

1. Limit groups are torsion-free ($\forall x \ (x \neq 1 \to x^n \neq 1)$).
2. Limit groups are commutative transitive ($\forall x \forall y \forall z \ ([x,y] = 1 \wedge [y,z] = 1 \to [x,z] = 1)$).
3. Any two elements of a limit group generate one of the following groups: $\{1\}, \mathbb{Z}, \mathbb{Z}^2, F_2$.
4. Limit groups are hyperbolic relative to free abelian groups (Dahmani).
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Generalizations to non-free settings

Another active research topic is limit groups over non-free groups. Some more (and less) recent work about limit groups over different classes of groups:

1. Torsion-free hyperbolic (Sela) and hyperbolic (André) groups.
2. Toral relatively hyperbolic groups (Kharlampovich-Miyasnikov, Groves).
3. Acylindrically hyperbolic groups (Groves-Hull, André-F).
4. (Coherent) RAAGs (Casals-Ruiz-Duncan-Kazachkov).


Thanks!
Questions?