Subgroup Convergence in Generalized Lamplighter Groups

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Any such triple defines a subgroup (via $\langle V_0, (v, s)\rangle$) and two triples (s, V_0, v) and (s', V_0', v') define the same subgroup if and only if $s = s'$, $V_0 = V_0'$, and $v + V_0 = v' + V_0$.

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$$V = (s, V_0, v) = \langle V_0, (v, s)\rangle < \mathcal{L}_n$$

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$$V_j \to V \quad \Longleftrightarrow \quad \mathbb{1}_{V_j} \to \mathbb{1}_V \text{ pointwise}$$

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If $V_j = (s_j, V_{0_j}, v_j) \longrightarrow V = (s, V_0, v)$ as subgroups of $\mathcal{L}_n$, do the terms

$s_j, V_{0_j},$ and v_j converge, respectively, to $s, V_0,$ and v in some sense?

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Let G be a group and let $K, H, H_1, H_2, ... < G$ with $H_j \to H$ in $\mathcal{S}(G)$, i.e., $\mathbb{1}_{H_j}(x) \to \mathbb{1}_H(x)$ for all $x \in G$. Then $H_j \cap K \to H \cap K$.

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Thus if $V_j = (s_j, V_{0_j}, v_j) \longrightarrow V = (s, V_0, v)$, then $V_{0_j} \to V_0$.

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References

R. Grigorchuk, R. Kravchenko (2012)
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Thank You!