Rapidly Rotating Convection with Nonlinear EoS

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Funded by: ERC, PCTS
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Louis Couston
BAS/DAMTP
How does convection interact with stable stratification?
Radiative zone = stably stratified

Diagram by Ben Brown
Earth’s Atmosphere

\[ \delta T \]

Stratosphere

Troposphere

Alexander & Barnet (2006)
Earth’s Atmosphere

12 km

Troposphere

400 km

Renaud et al (2019)

Alexander & Barnet (2006)
Earth’s Atmosphere

Renaud et al (2019)

Troposphere

Alexander & Barnet (2006)
Equation of State

Normal Boussinesq liquid

slope is $-\alpha$
Equation of State of water
Equation of State of water

Stably stratified

Convective
Water Experiment

Le Bars et al. 2015

Dimensions: 20 x 4 x 35 cm³
Ra ~ 2x10⁷ - 2x10⁸
Water Experiment

Le Bars et al. 2015

Dimensions: 20 x 4 x 35 cm³
Ra ~ 2x10⁷ - 2x10⁸
Water Experiment

Le Bars et al. 2015

Dimensions: 20 x 4 x 35 cm³
Ra ~ 2x10⁷ - 2x10⁸
Equation of State of water
Nonlinear EoS

\[ \rho = \begin{cases} 
-\frac{T}{ST} & T \leq 0 \\
\frac{T}{ST} & T > 0 
\end{cases} \]
Nonlinear EoS

\[ \rho = \begin{cases} 
  -T & T \leq 0 \\
  ST & T > 0 
\end{cases} \]

- stably stratified
- convective
\partial_t u + \nabla p - Pr \nabla^2 u = -u \cdot \nabla u + RaPr \rho(T) e_z

\partial_t T - \nabla^2 T = -u \cdot \nabla T

\nabla \cdot u = 0

\rho = \left\{ \begin{array}{cc}
-T & T \leq 0 \\
ST & T > 0
\end{array} \right\
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The team so far

Daniel Lecoanet (Princeton)  Keaton Burns (MIT)
Jeff Oishi (Bates)       Ben Brown (Colorado)
Geoff Vasil (Sydney)
$t/\tau_c = 22137$

$\omega/\tau_c$

$\rho$

$x$

$z$
Lower Pr = v/κ

![Graph with x and z axes, color scale, and w values at t=2.18]
Lower $Pr = \nu / \kappa$

$t = 2.18$

$w$

$x$

$z$
Quasi-Biennial Oscillation

Equatorial Zonal Wind, Deseasoned Monthly Means

Baldwin+ 2001
Quasi-Biennial Oscillation

Baldwin+ 2001
\[ F_w \sim F_c \frac{1}{N\tau_c} (\omega\tau_c)^{-13/2} (k_\perp H)^4 \]

Couston et al (2018)
\[ F_w \sim F_c \frac{1}{N\tau_c} (\omega\tau_c)^{-13/2} (k_\perp H)^4 \]

\[ F_w(k_\perp) \]

\[ k_\perp H \]

Couston et al (2018)
Rapidly Rotating Convection
Rapidly Rotating Convection

Stellmach et al (2014)
Rapidly Rotating Convection

Stellmach et al (2014)
Rapidly Rotating Convection

\[ u^2 + v^2 \]

stress-free!!

\[ \omega_z \]

Stellmach et al (2014)
Figure 1: Meridional and equatorial cross-sections of a snapshot of the axial vorticity in the 3D model for $E_k = 10^{8}$, $R_a = 2 \times 10^{10}$, and $Pr = 10^{2}$. Streamlines have been superimposed in the equatorial plane. The kinetic energy of the velocity projected on a quasi-geostrophic state ($u_s, u, u_s$) (where the angle brackets denote an axial average) is within 0.2% of the total kinetic energy.

For the low Ekman numbers studied here, convection is always in a turbulent state, even near onset, $Re \approx 10^{3}$. The convection takes the form of vortical plumes that are radially elongated on scales much shorter than the outer radius (Figure 2). At large radius, the steepening of boundary slope leads to rapid changes in the column height, which inhibit convection.

The dynamics there mainly consists of Rossby waves, which appear as elongated vortices with a prograde tilt (Figure 2e). Their radial velocity is relatively small so conduction dominates the heat transport in the outer part of the equatorial plane.

Hereafter we solely consider the dynamics of the inner convective region, which grows wider with increasing Rayleigh number ($R_a$, which measures the strength of the buoyancy driving with respect to dissipative effects). The azimuthal lengthscale of the convective flows decreases notably with radius (Figures 2f-g) to minimise the changes in the column height. At lower $E_k$, the scale of the convective flow is visibly smaller. We find that the convective lengthscale is controlled by the Rossby number, rather than by any viscous effect. The flows shown in Figures 1 and 2 are snapshots taken once the system has reached a statistically steady state, and are entirely unlike the linear viscous mode at the convection onset, which consists of drifting columns with a narrow azimuthal lengthscale that scales as $E_k^{-1/3}$.

15, 16 The convective lengthscale increases with the buoyancy driving as seen on the power spectra of the total and radial kinetic energies in Figure 3. The peak of the radial kinetic energy moves to smaller wavenumber for increasing $R_a$, as can be observed for the two different Rayleigh numbers shown at $E_k = 10^{8}$, and is $2 \times 10^{10}$.
Rapidly Rotating Convection

Guervilly et al (2018)

Figure 2: Snapshots of the radial velocity in a quarter of the equatorial plane during the statistically steady phase for a) $E_k = 10^8$, $R_a = 2.5 \times 10^{10}$ (3D model), b) $E_k = 10^9$, $R_a = 2.7 \times 10^{11}$ (QG model), c) $E_k = 10^{10}$, $R_a = 6.3 \times 10^{12}$ (QG model) and d) $E_k = 10^{11}$, $R_a = 5.25 \times 10^{13}$ (QG model). Close-ups of the equatorial plane are shown in e-g for the same parameters as in d; e shows the outer conduction-dominated region where the dynamics is dominated by Rossby waves, and f-g the inner convective region. The Prandtl number is $10^{2}$ in all cases. The colorbars give the radial velocity normalised by the viscous velocity scale (i.e. corresponding to a Reynolds number).
Rapidly Rotating Convection

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What sets the lengthscale??

Guervilly et al (2018)
Rapidly Rotating Convection

Favier et al. (2019)

Figure 1. (a) Time evolution of the kinetic energy density $K_2 D$ for different initial vortex amplitudes $A$. The grey area corresponds to the transition where $K_2 D \approx K_3 D$. (b) Time evolution of the ratio between $K_2 D$ and the total kinetic energy $K_2 D + K_3 D$ for different amplitudes $A$. Lines in figure 1(a), indicating that there is no significant energy transfer from the 3D fluctuations to the 2D flow. For larger amplitudes however, typically $A \approx 800$, we observe an initial decay of $K_2 D$ followed by an approximately linear increase until the energy eventually saturates at very long times. Note that close to the transition threshold, see case $A = 875$ for example, it is not yet clear whether the vortex will grow or decay. In view of figure 1(a), which shows that the large-scale vortex has not

Figure 2. Visualizations of the quasi-steady states at $t = 0$. (a, b) Vertical vorticity component. (c, d) Velocity field streamlines coloured with the velocity amplitude. (a, c) No initial vortex dipole $A = 0$. (b, d) Initial vortex dipole amplitude $A = 1600$. 

$\omega_z$

$u$

$\times 10^4$
Rapidly Rotating Convection

Rapidly Rotating Convection

1. Require stress-free BC’s
2. Inverse cascade to box size
Stratification
stress-free, no strat

FIG. 1. Volume rendering of the horizontal velocity amplitude
Stratification

stress-free, no strat

no-slip, no strat

The results are shown at the steady state for simulations (A state). In fully-convective simulations without a stable layer, a LSV emerges when the top boundary is free-slip (figure 1A), even with a no-slip top boundary (cf. one LSV in figure 1B and several smaller LSVs in figure 1C). This means that stable layers protect upscale energy transfers and LSVs against boundary friction, which is a fundamental result for planetary cores and potentially for Earth's oceans and subsurfaces. It shows that LSVs are expected to be robust against no-slip boundaries in reduced models of fully-convective fluids assuming subadiabatic layers of planetary cores and oceans' pycnoclines can play an important role in the dynamics of LSVs. However, LSVs cannot emerge in a convective fluid directly adjacent to a no-slip bottom boundary for our choice of parameters. We recall that the bottom boundary is free-slip in all our simulations, since it may not be always necessary, but it still broadens the domain of existence of LSVs to cases accessible to DNS and asymptotically-large rotation and turbulence intensity. Therefore, a stable layer tampering boundary friction through a stably-stratified layer provides a natural saturation mechanism for LSVs, and that the boundary friction inhibits upscale energy transfers in fully-convective fluids, to the point that, as shown by previous studies, large-scale barotropic vortices cannot be obtained in current DNS (i.e. which are limited to relatively high viscosity) with no-slip boundaries. With a stable layer (figure 1A), but not when the top boundary is no-slip (cf. figure 1A). Thus, boundary friction inhibits upscale energy transfers in fully-convective fluids, to the point that, as shown by previous studies, large-scale barotropic vortices cannot be obtained in current DNS (i.e. which are limited to relatively high viscosity) with no-slip boundaries. With a stable layer, the LSV is wide and weakly-penetrating (B) the LSV is wide and weakly-penetrating while in (C) there are several tall LSVs that penetrate far into the stable fluid. This means that stable layers protect upscale energy transfers and LSVs against boundary friction, which is a fundamental result for planetary cores and potentially for Earth's oceans and subsurfaces. It shows that LSVs are expected to be robust against no-slip boundaries in reduced models of fully-convective fluids assuming subadiabatic layers of planetary cores and oceans' pycnoclines can play an important role in the dynamics of LSVs. However, LSVs cannot emerge in a convective fluid directly adjacent to a no-slip bottom boundary for our choice of parameters. We recall that the bottom boundary is free-slip in all our simulations, since it may not be always necessary, but it still broadens the domain of existence of LSVs to cases accessible to DNS and asymptotically-large rotation and turbulence intensity. Therefore, a stable layer tampering boundary friction through a stably-stratified layer provides a natural saturation mechanism for LSVs, and that the boundary friction inhibits upscale energy transfers in fully-convective fluids, to the point that, as shown by previous studies, large-scale barotropic vortices cannot be obtained in current DNS (i.e. which are limited to relatively high viscosity) with no-slip boundaries. With a stable layer, the LSV is wide and weakly-penetrating (B) the LSV is wide and weakly-penetrating while in (C) there are several tall LSVs that penetrate far into the stable fluid. This means that stable layers protect upscale energy transfers and LSVs against boundary friction, which is a fundamental result for planetary cores and potentially for Earth's oceans and subsurfaces. It shows that LSVs are expected to be robust against no-slip boundaries in reduced models of fully-convective fluids assuming subadiabatic layers of planetary cores and oceans' pycnoclines can play an important role in the dynamics of LSVs.
Stratification

stress-free, no strat

no-slip, no strat

stratified layer

no-slip

stress-free

A. Importance of stably-stratified layers

In fully-convective simulations without a stable layer, a LSV emerges when the top boundary is free-slip (figure 1A), even with a no-slip top boundary (cf. one LSV in figure 1B and several smaller LSVs in figure 1C). This shows that LSVs cannot emerge in a convective fluid directly adjacent to a no-slip bottom boundary for our choice of parameters.

With stratification, LSVs are expected to be robust against no-slip boundaries in reduced models of fully-convective fluids assuming asymptotically-large rotation and turbulence intensity [42]. Therefore, a stable layer tampering boundary friction means that stable layers protect upscale energy transfers and LSVs against boundary friction, which is a fundamental important result for planetary cores and potentially for Earth’s oceans and subsurfaces oceans. It shows that subadiabatic layers of planetary cores and oceans’ pycnoclines can play an important role in the dynamics of LSVs and may not be always necessary, but it still broadens the domain of existence of LSVs to cases accessible to DNS and possibly laboratory experiments [43].

B. Horizontal saturation

When a stable layer is present, the horizontal velocity amplitude $v$ is large in the region $-1 < W < 1$ of table I. Dark (light) blue colors highlight the turbulent (stable) fluid region in (B-C). In (B), the LSV is wide and weakly-penetrating while in (C) there are several tall LSVs that penetrate far into the stable fluid.

We first show in figure 1 three-dimensional snapshots of the horizontal velocity amplitude $V(u^2 + v^2)$ at steady state. The orange colors show large (small) velocities. Dark (light) blue colors highlight the turbulent (stable) fluid region in (B-C). In figure 1A, the stratified layer is stress-free, no strat, while in figure 1B, the stratified layer is no-slip, no strat.

III. RESULTS

A. Importance of stably-stratified layers

We find that one or several LSVs always emerge for the same convective parameters as in Figure 1, even with a no-slip top boundary (cf. one LSV in Figure 1B and several smaller LSVs in Figure 1C). This means that stable layers protect upscale energy transfers and LSVs against boundary friction, which is a fundamental important result for planetary cores and potentially for Earth’s oceans and subsurfaces oceans. It shows that subadiabatic layers of planetary cores and oceans’ pycnoclines can play an important role in the dynamics of LSVs and may not be always necessary, but it still broadens the domain of existence of LSVs to cases accessible to DNS and possibly laboratory experiments [43].

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FIG. 2. (A) Integral length scale $L_0$ (proxy for LSV diameter) in the middle of the convection zone as a function of time $t$ for the simulations of Table I. Blue, green, orange colors denote weak, moderate, strong stratifications, and thicker lines correspond to thicker stable fluid layers. Dashed lines indicate results obtained with a free-slip condition on the top boundary and the dotted line shows the box size $L = 4$. (B-D) Snapshots of horizontal velocity $V$ in the middle of the convection zone at steady-state for simulations $M_2$, $M_1$, $M_0.5$. Of the vertical temperature gradient with radius results in a negative temperature anomaly, $T_0 = T_1$, in the LSV centre. This anomaly is shown by the light red-coloured cone in figure 3A and is small, as is the buoyancy anomaly $b_0 = T_0 < 0$, in most of the convective layer. As a result, the LSV roughly satisfies the Taylor-Proudman theorem, i.e. is depth-invariant, in the convective layer (cf. equation (3)). The negative temperature anomaly increases with height, such that at and above the base of the stably-stratified layer, it translates into a positive and potentially large buoyancy anomaly $b_0 = ST_0$. This positive buoyancy anomaly drives the decay of the azimuthal velocity with height above the black dashed line according to the thermal wind balance (equation (3)), which is why the stratified LSV has a half-dome shape. When $S$ increases, i.e. the stratification becomes stronger, $b_0$ increases, such that the aspect ratio $h/\lambda$ of a LSV must decrease in order to satisfy the thermal wind balance. This explains why in a strongly-stratified fluid LSVs appear as wide weakly-penetrating columns (cf. figure 1B), while in a weakly-stratified fluid they appear as tall narrow columns (figure 1C).

Figures 3B,C show the vertical vorticity for simulations $M_2$ (figure 3B) and $M_0.5$ (figure 3C), i.e. which have a deep and shallow stratified layer, respectively, but same parameters otherwise. As described above, the stratified vortex cap has a positive buoyancy anomaly in both cases (as shown by the gray contours), which is balanced by a vorticity decay with height above the convective-stable interface (dashed line). However, while the penetration of the vortex cap is small compared to the stable layer thickness $H$ in figure 3B, the penetration is large enough compared to $H$ in figure 3C such that the LSV is confined vertically. The maximum vorticity does not change significantly between the two simulations and the buoyancy anomaly is smaller in figure 3C than in figure 3B (cf. in-line numbers). Thus, $|@z v|$ is larger for a vertically-confined LSV than for a vertically-unconfined LSV, which means that confined LSVs must decrease in diameter (compared to their unconfined counterparts), i.e. such that $|@r b_0|$ increases, in order to maintain thermal wind balance. As a result, boundary friction makes the LSVs saturate naturally in general and in particular in figure 3C, because it imposes a sharp vorticity decay that can only be balanced by a reduction of the LSV diameter. It can be noted that the horizontal narrowing of vertically-confined LSVs does not apply when the top boundary is free-slip since in this case the vorticity doesn't decay any quicker than when it is unconfined.
This anomaly is shown by the light red-coloured cone in figure 3A and is small, as is the buoyancy anomaly. Thus, between the two simulations and the buoyancy anomaly is smaller in figure 3C than in figure 3B (cf. in-line numbers).

The vortex cap is small compared to the stable layer thickness. The vortex cap has a positive buoyancy anomaly in both cases (as shown by the gray contours), which is balanced by a half-dome shape. When above the black dashed line according to the thermal wind balance (equation (3)), which is why the stratified LSV has a half-dome shape. When above the black dashed line according to the thermal wind balance (equation (3)), which is why the stratified LSV has a half-dome shape.

Because of this, the LSV must decrease in diameter (compared to their unconfined counterparts), i.e. such that the LSV is confined vertically. The maximum vorticity does not change significantly.

As a result, boundary friction makes the LSVs saturate naturally in general and, in particular in figure 3C, because it imposes a sharp vorticity decay that can only be balanced by a reduction of the LSV diameter. It can be noted that the horizontal narrowing of vertically-confined LSVs does not apply when the top boundary is free-slip since in this case the vorticity doesn't decay any quicker than when it is unconfined.

In a strongly-stratified fluid, buoyancy is strongly balanced by vertical exchange, and thus, the LSVs must decrease in order to satisfy the thermal wind balance. This explains why in a strongly-stratified fluid LSVs appear as wide weakly-penetrating columns (cf. figure 1B), while in a weakly-stratified fluid they appear as tall narrow columns (figure 1C).

FIG. 2. (A) Integral length scale \( L_M \) (proxy for LSV diameter) in the middle of the convection zone as a function of time \( t \). This positive buoyancy anomaly drives the decay of the azimuthal velocity with height such that at and above the base of the stably-stratified layer, it translates into a positive and potentially large

The vertical temperature gradient with radius results in a negative temperature anomaly, such that the aspect ratio increases, in order to maintain thermal wind balance. As a result, boundary friction makes the LSVs saturate naturally in general and, in particular in figure 3C, because it imposes a sharp vorticity decay that can only be balanced by a reduction of the LSV diameter. It can be noted that the horizontal narrowing of vertically-confined LSVs does not apply when the top boundary is free-slip since in this case the vorticity doesn't decay any quicker than when it is unconfined.

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This anomaly is shown by the light red-coloured cone in figure 3A and is small, as is the buoyancy anomaly.

Thus, the vortex cap has a positive buoyancy anomaly in both cases (as shown by the gray contours), which is balanced by a deep and shallow stratified layer, respectively, but same parameters otherwise. As described above, the stratified vortex is taller and thinner than the unstratified one (figure 3C), i.e. which have a half-dome shape. When the top boundary is free-slip since in this case the vorticity doesn't decay any quicker than when it is unconfined.

However, while the penetration of the fluid LSVs appear as wide weakly-penetrating columns (cf. figure 1B), while in a weakly-stratified fluid they appear as tall narrow columns (figure 1C).

Figures 3B,C show the vertical vorticity for simulations in figure 3C such that the LSV is confined vertically. The maximum vorticity does not change significantly.

The buoyancy anomaly is depth-invariant, in the convective layer (cf. equation (3)). The negative temperature anomaly increases with distance between the two simulations and the buoyancy anomaly is smaller in figure 3C than in figure 3B (cf. in-line numbers).

The horizontal velocity at steady-state for simulations corresponds to thicker stable fluid layers. Dashed lines indicate results obtained with a free-slip condition on the top boundary.

The temperature anomaly, i.e. is depth-invariant, in the convective layer (cf. equation (3)), which is why the stratified LSV has a positive buoyancy anomaly.

This positive buoyancy anomaly drives the decay of the azimuthal velocity with height, such that at and above the base of the stably-stratified layer, it translates into a positive and potentially large negative temperature anomaly.

The aspect ratio increases, i.e. the stratification becomes stronger, such that the aspect ratio increases, in order to maintain thermal wind balance. As a result, boundary friction makes the LSVs saturate naturally in general and the LSV diameter.

It can be noted that the horizontal narrowing of vertically-confined LSVs does not apply when the LSV is confined vertically.

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Vortex Cap
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]
Vortex Cap

\[ \partial_r p' =fv_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \oint_C \nabla p' \cdot d\ell = 0 \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \oint_C \nabla p' \cdot d\ell = 0 \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \int_C \nabla p' \cdot d\ell = 0 \]

\[ \int_d \nabla p' \cdot d\ell = \int f v_\theta dr > 0 \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \oint_c \nabla p' \cdot d\ell = 0 \]

\[ \oint_d \nabla p' \cdot d\ell = \int f v_\theta dr > 0 \]

\[ \oint_c \nabla p' \cdot d\ell = - \int b' dz \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \oint_c \nabla p' \cdot d\ell = 0 \]

\[ \oint_d \nabla p' \cdot d\ell = \int f v_\theta dr > 0 \]

\[ \oint_{c'} \nabla p' \cdot d\ell = - \int b' dz < 0 \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \int_C \nabla p' \cdot d\ell = 0 \]

\[ \int f v_\theta dr = \int b' dz \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \oint_C \nabla p' \cdot d\ell = 0 \]

\[ \int f v_\theta dr = \int b' dz \]

\[ \ell f v_\theta \sim hN^2 \delta \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \ell f v_\theta \sim h N^2 \delta \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \ell f v_\theta \sim h N^2 \delta \]

\[ Ro = \frac{v_\theta}{f \ell} \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ \ell f v_\theta \sim h N^2 \delta \]

\[ \frac{\ell^2}{h \delta} \sim Ro \frac{N^2}{f^2} \]

\[ b' = b - b_\infty \]

\[ Ro = \frac{v_\theta}{f \ell} \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

\[ b' = b - b_\infty \]

\[ \ell f v_\theta \sim h N^2 \delta \]

\[ \frac{\ell^2}{h \delta} \sim Ro \frac{N^2}{f^2} \]

\[ \frac{\ell}{h} \sim \frac{N}{f} \sqrt{Ro} \]
Vortex Cap

\[ \partial_r p' = f v_\theta \quad \partial_z p' = b' \]

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FIG. 3. (A) Schematic of the axisymmetric structure of LSVs obtained in DNS with $h$ and $\ell$ the LSV diameter, penetration depth and restratification depth. The stratified vortex cap is the part of the LSV that is above the convective-stable interface (black dashed line) and is highlighted by a solid white line. The red cone highlights the region where the temperature anomaly $T_0 < 0$. (B-C) Map of vertical vorticity $\epsilon_{11}/Pr$ in a cylindrical coordinates system centred on the vortex core after time and azimuthal averaging for simulations $M_{2}$ and $M_{0.5}$, respectively (cf. table I). The solid lines with grey color scale show isocontours of buoyancy anomaly $b_0 > 0$. The red solid line is a contour of constant vorticity.

D. Aspect ratio of the stratified vortex cap

FIG. 4. (A) Aspect ratio $h/\ell$ of the stratified vortex cap against the theoretical prediction (5) for $\alpha$. (B) LSV radius $\ell$ as a function of $H/\ell$. The solid line shows that the radius of saturated LSVs follows the same trend as the maximum radius $\ell_{\text{max}} = H/(2\alpha)$ predicted for LSVs that are confined vertically. LSVs that are not confined vertically have $\ell < \ell_{\text{max}}$ and saturate at the box size (cf. three rightmost symbols shown as circles).

The aspect ratio of the stratified vortex cap, $\alpha = h/\ell$, is a function of the normalized stratification strength $N/f$, with $N = f\epsilon_{11}N/Pr$ the dimensional buoyancy frequency, and the Rossby number of the LSV, i.e. $Ro = \epsilon_{11}(v_0\sqrt{f}/Pr)/\ell$ with $v_0$ the maximum azimuthal velocity at the base of the stratified vortex cap. An approximate expression for $\alpha (Ro, N/f)$ can be derived from the hydrostatic and cyclo-geostrophic equations, which are slightly more relevant in

$$\frac{\ell}{h} \sim \frac{N}{f} \sqrt{Ro}$$
\[
\frac{\ell}{h} \sim \frac{N}{f} \sqrt{Ro}
\]
\[ \frac{\ell}{h} \sim \frac{N}{f} \sqrt{Ro} \]

Vortex Cap

A

B

\( h < 2 \)
\[ \frac{h}{\ell} \sim \frac{N}{f} \sqrt{Ro} \]

Vortex Cap

\( h < 2 \)

\( h > \frac{1}{2} \)
FIG. 3. (A) Schematic of the axisymmetric structure of LSVs obtained in DNS with \( h \) and \( \ell \) the LSV diameter, penetration depth and restratification depth. The stratified vortex cap is the part of the LSV that is above the convective-stable interface (black dashed line) and is highlighted by a solid white line. The red cone highlights the region where the temperature anomaly \( T_0 < 0 \). (B-C) Map of vertical vorticity \( \varepsilon_k \) in a cylindrical coordinates system centred on the vortex core after time and azimuthal averaging for simulations \( M_2 \) and \( M_0 \) respectively (cf. table I). The solid lines with grey color scale show isocontours of buoyancy anomaly \( b_0 > 0 \). The red solid line is a contour of constant vorticity.

D. Aspect ratio of the stratified vortex cap

FIG. 4. (A) Aspect ratio \( h/\ell \) of the stratified vortex cap against the theoretical prediction (5) for \( \varepsilon_k \). (B) LSV radius \( \ell \) as a function of \( H/\ell \). The solid line shows that the radius of saturated LSVs follows the same trend as the maximum radius \( \ell_{\text{max}} = H/(2\varepsilon_k) \) predicted for LSVs that are confined vertically. LSVs that are not confined vertically have \( \ell < \ell_{\text{max}} \) and saturate at the box size (cf. three rightmost symbols shown as circles). The aspect ratio of the stratified vortex cap, \( \varepsilon_k = h/\ell \), is a function of the normalized stratification strength \( N/f \), with \( N = f \varepsilon_k N/Pr \) the dimensional buoyancy frequency, and the Rossby number of the LSV, i.e. \( Ro = \varepsilon_k (v_0 \sqrt{Pr})/\ell \) with \( v_0 \) the maximum azimuthal velocity at the base of the stratified vortex cap. An approximate expression for \( \varepsilon_k (Ro, N/f) \) can be derived from the hydrostatic and cyclo-geostrophic equations, which are slightly more relevant in \( \varepsilon_k \sim N/f \).
FIG. 3. (A) Schematic of the axisymmetric structure of LSVs obtained in DNS with $h$ and the LSV diameter, penetration depth and restratification depth. The stratified vortex cap is the part of the LSV that is above the convective-stable interface (black dashed line) and is highlighted by a solid white line. The red cone highlights the region where the temperature anomaly $T_0 < 0$. (B-C) Map of vertical vorticity $\varepsilon$ in a cylindrical coordinates system centred on the vortex core after time and azimuthal averaging for simulations $M_2$ and $M_0$.5, respectively (cf. table I). The solid lines with grey color scale show isocontours of buoyancy anomaly $b_0 > 0$. The red solid line is a contour of constant vorticity.

D. Aspect ratio of the stratified vortex cap

FIG. 4. (A) Aspect ratio $h/\ell$ of the stratified vortex cap against the theoretical prediction (5) for $\varepsilon$. (B) LSV radius $\ell$ as a function of $H/\varepsilon$. The solid line shows that the radius of saturated LSVs follows the same trend as the maximum radius $\ell_{\text{max}} = H/(2\varepsilon)$ predicted for LSVs that are confined vertically. LSVs that are not confined vertically have $\ell < \ell_{\text{max}}$ and saturate at the box size (cf. three rightmost symbols shown as circles).

The aspect ratio of the stratified vortex cap, $\varepsilon = h/\ell$, is a function of the normalized stratification strength $N/f$, with $N = f\varepsilon N/P_r$ the dimensional buoyancy frequency, and the Rossby number of the LSV, i.e. $Ro = \varepsilon (v_0 \sqrt{f}/P_r)$/\ell$. An approximate expression for $(\varepsilon, N/f, \varepsilon)$ can be derived from the hydrostatic and cyclo-geostrophic equations, which are slightly more relevant in $\varepsilon \ll N/f$. Fix ell. Then height of vortex cap is $\alpha \ell$.

1. Fix ell. Then height of vortex cap is $\alpha \ell$.
FIG. 3. (A) Schematic of the axisymmetric structure of LSVs obtained in DNS with $\ell$, $h$ and the LSV diameter, penetration depth and restratification depth. The stratified vortex cap is the part of the LSV that is above the convective-stable interface (black dashed line) and is highlighted by a solid white line. The red cone highlights the region where the temperature anomaly $T_0 < 0$. (B-C) Map of vertical vorticity $\vec{e}_k/Pr$ in a cylindrical coordinates system centred on the vortex core after time and azimuthal averaging for simulations $M_2$ and $M_0$.5, respectively (cf. table I). The solid lines with grey color scale show isocontours of buoyancy anomaly $b_0 > 0$. The red solid line is a contour of constant vorticity.

D. Aspect ratio of the stratified vortex cap

FIG. 4. (A) Aspect ratio $h/\ell$ of the stratified vortex cap against the theoretical prediction $(5)$ for $\ell_0$. (B) LSV radius $\ell$ as a function of $H/\ell$. The solid line shows that the radius of saturated LSVs follows the same trend as the maximum radius $\ell_{\text{max}} = H/(2\ell)$ predicted for LSVs that are confined vertically. LSVs that are not confined vertically have $\ell < \ell_{\text{max}}$ and saturate at the box size (cf. three rightmost symbols shown as circles).

The aspect ratio of the stratified vortex cap, $\ell = h/\ell$, is a function of the normalized stratification strength $N/f$, with $N = f\vec{e}_k N/Pr$ the dimensional buoyancy frequency, and the Rossby number of the LSV, i.e. $Ro = \vec{e}_k (v_0 \bar{\ell}/Pr)/\ell$ with $v_0 \bar{\ell}$ the maximum azimuthal velocity at the base of the stratified vortex cap. An approximate expression for $\ell (Ro, N/f)$ can be derived from the hydrostatic and cyclo-geostrophic equations, which are slightly more relevant in $\vec{e}_k \ll N/f$.

1. Fix ell. Then height of vortex cap is $\alpha \ell$

2. Fix maximum height $H$. Then vortex will saturate at $\ell \sim H/\alpha$
Conclusions

1. Stratified layer above rapidly rotating convection -> LSV w/ no-slip BCs

2. Stratified layer can saturate LSV size

3. Aspect ratio \( \frac{\ell}{h} \sim \frac{N}{f} \sqrt{\text{Ro}} \)