Low Mach Number Modeling: Algorithms and Applications

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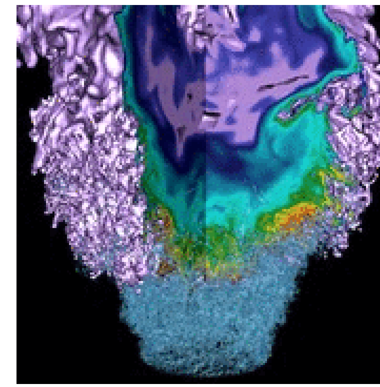
Motivation

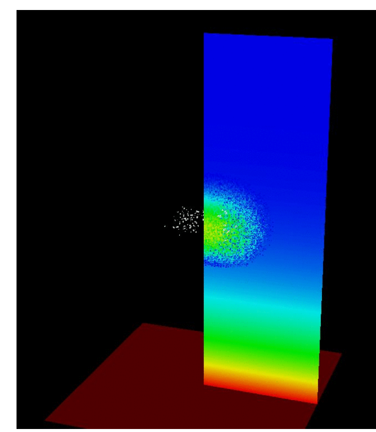
Many fluid flows of interest include processes that occur at very different spatial and temporal scales. Specifically, if the fluid velocity is much less than the sound speed, we can characterize them as “**low Mach number flows**”

- Combustion (M ~ 0.01)
  - mean flow $O(10^3)$ cm/s (flame speed ~ 10^2 cm/s)
  - sound speed $O(10^5)$ cm/s



- Atmosphere (M < 0.20)
  - a Category 5 hurricane has wind speeds ~ 70 m/s;
  - sound speed in air ~ 340 m/s

- Ocean (M < 0.004)
  - Gulf Stream ~ 6 m/s
  - sound speed in water c ~ 1500 m/s,



- Stellar convection (M ~ 0.02)
  - Gas velocity ~ 1e5 m/2
  - Sound speed in hot gas ~ 5e6 m/s

Low Mach Number Modeling

Let's start with a simple system of compressible flow equations –

- no reactions
- no gravity
- gamma-law gas

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0$$ Conservation of mass

$$\frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U U + p) = 0$$ Conservation of momentum

$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho U E + U p) = 0$$ Energy equation

$$p = (\gamma - 1)\rho e$$ Equation of state (EOS)

The coupling between p and ρ enables the propagation of acoustic waves.

We want to develop a model that doesn't allow acoustic wave propagation, so we need to somehow decouple pressure and density.

The essence of low Mach number modeling

We want to develop a model based on separation of scales between fluid motion and acoustic wave propagation.

One approach is based on asymptotic expansion in the (small) Mach number, M, which leads to a decomposition of the pressure into thermodynamic and dynamic components:

$$\boxed{p = p_0 + p'}$$

where $p' / p_0 = O(M^2)$

- p_0 replaces p in the thermodynamic equation
- p' appears only in the momentum equation

Low Mach Number Modeling

We can replace our compressible equation set –

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) &= 0 \\
\frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U U + p) &= 0 \\
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho U E + U p) &= 0 \\
p &= (\gamma - 1)\rho e
\end{aligned}
$$

by its low Mach number counterpart:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) &= 0 \\
\frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U U + p') &= 0 \\
\nabla \cdot U &= 0 \\
p_0 &= (\gamma - 1)\rho e
\end{aligned}
$$

Low Mach Number Modeling

Where did the divergence constraint on velocity come from?

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) &= 0 \\
\frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U U + p') &= 0 \\
\nabla \cdot U &= 0 \\
p_0 &= (\gamma - 1)\rho e
\end{aligned}
$$

We take the material derivative of the EOS to get an expression relating the divergence of velocity to the time evolution of the other quantities.

In this simplest case, $\nabla \cdot U = 0$ is the statement of how to keep the pressure on the constraint $p = p_0$ where p_0 is constant in space and time.

Low Mach Number Models

- **Physically**: acoustic equilibration is instantaneous; sound waves are "filtered" out
- **Mathematically:** resulting equation set is no longer strictly hyperbolic; a constraint equation is added to the evolution equations
- **Computationally:** explicit time step is dictated by fluid velocity, not sound speed.

$$\boxed{\Delta t_{\mathrm{LM}} < \frac{\Delta x}{u},} \quad \Delta t_{\mathrm{C}} < \frac{\Delta x}{u + c}$$

A Visual Interpretation of a Low Mach Number Model

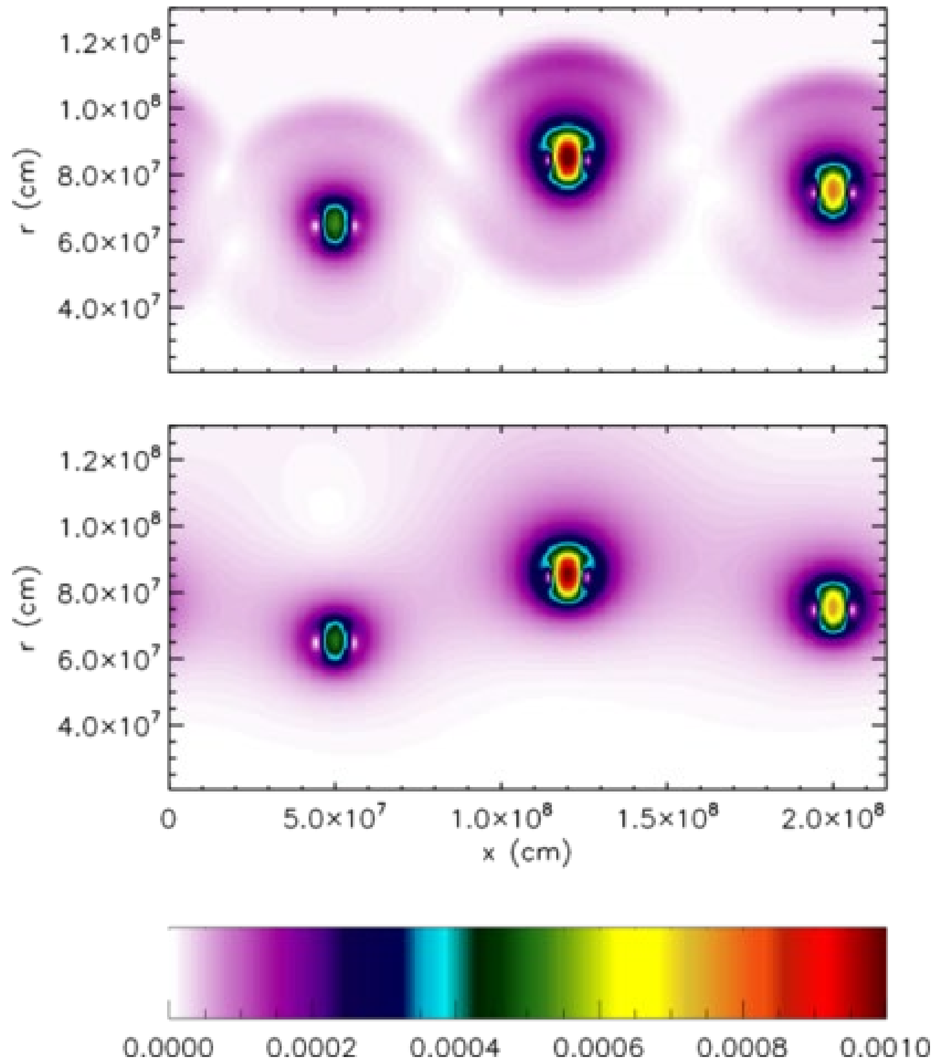
Mach number for 3 hot bubbles rising in a white dwarf environment

- Top (compressible)
- Bottom (low Mach)

We observe:

- The compressible flow has a finite speed of propagation; the LM flow instantly equilibrates

- The dynamics of the bubble itself – the shape and speed of rise – are indistinguishable between the two cases



Credit to Mike Zingale

A bit of history …

Low-speed models for combustion based on low Mach number asymptotics are relatively well established, going back to Majda&Sethian (1985); Day & Bell (2000) and many more

"Soundproof models" are also well established in atmospheric science – in particular

- **anelastic** (Batchelor(1953); Ogura&Philips (1962), Dutton&Fichtl (1969), Gough (1969))
- **pseudo-incompressible** models (Durran 1989)

But how to generalize these models to stratified reacting astrophysical flows?

And why would you want to?

Most astrophysical phenomena are explosive

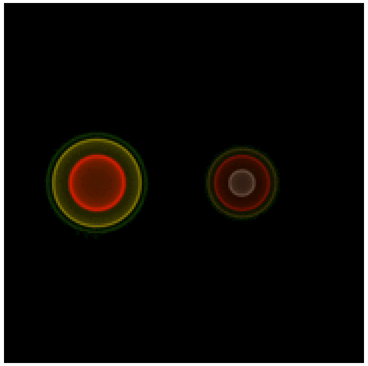
Most astrophysical flows of interest are explosive, and we typically want to simulate the explosion:

Whole star:

- supernovae (largest thermonuclear explosion = death of a star)
- gamma-ray bursts (brightest electromagnetic events in universe; from collapse of star or possibly merger of binary neutron stars)

Layer:

- classical novae (burst from accreted H/He layer on white dwarf)
- X-ray bursts (bursts from accreted He layer on neutron star)



Credit to Max Katz

The problem is that the simulated explosions can be very sensitive to the initial conditions

And while the explosion of a Type Ia supernova may be over in 2 seconds, understanding the evolution of the progenitor requires simulation of at least hours

Compressible astrophysics

Standard astrophysical models add reactions, self-gravity and a complicated equation of state to our standard Euler equations.

$$\begin{aligned}
\frac{\partial(\rho X_k)}{\partial t} + \nabla \cdot (\rho U X_k) &= \rho \dot{\omega}_k && \text{Conservation of mass} \\
\frac{\partial(\rho U)}{\partial t} + \nabla \cdot (\rho U U + p) &= -\rho g \mathbf{e}_r && \text{Conservation of momentum} \\
\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho U E + U p) &= -\rho g (U \cdot \mathbf{e}_r) + \rho \sum_k q_k \dot{\omega}_k && \text{Energy equation}
\end{aligned}$$

ρ	density	e	internal energy
U	velocity	X_k	mass fractions
p	pressure	$\dot{\omega}_k$	X_k production rate
$E = e + U^2/2$	total energy	$\vec{g}$	gravity

with Timmes equation of state:

$$p(\rho, T, X_k) = p_{ele} + p_{rad} + p_{ion} \qquad \text{Equation of State (EOS)}$$

Can we use one of the Low Mach Number Models we already have?

We need a model that is general enough to keep all of the features we need for a star to get ready to explode:

- Background stratification
- Large deviations of density and temperature from background state
- Reactions with heat release
- Nonideal equation of state
- Overall expansion of the star (i.e. p_0 can change in time)

but not allow the propagation of acoustic waves

And the answer is …. no

There is a hierarchy of existing models …

X

- Constant density **incompressible**
- Variable density incompressible
- **Boussinesq**

$$\nabla \cdot U = 0$$

No background stratification or reactions

X

- **(Linearized) Anelastic**
  - Compressibility due to stratification
  - Small thermodynamic perturbations

$$\nabla \cdot (\rho_0 U) = 0$$

Assumes density close to ambient

X

- **Low Mach number combustion**
  - Local compressibility due to heat release
  - Large thermodynamic perturbations ok

$$\nabla \cdot U = S$$

No background stratification

X

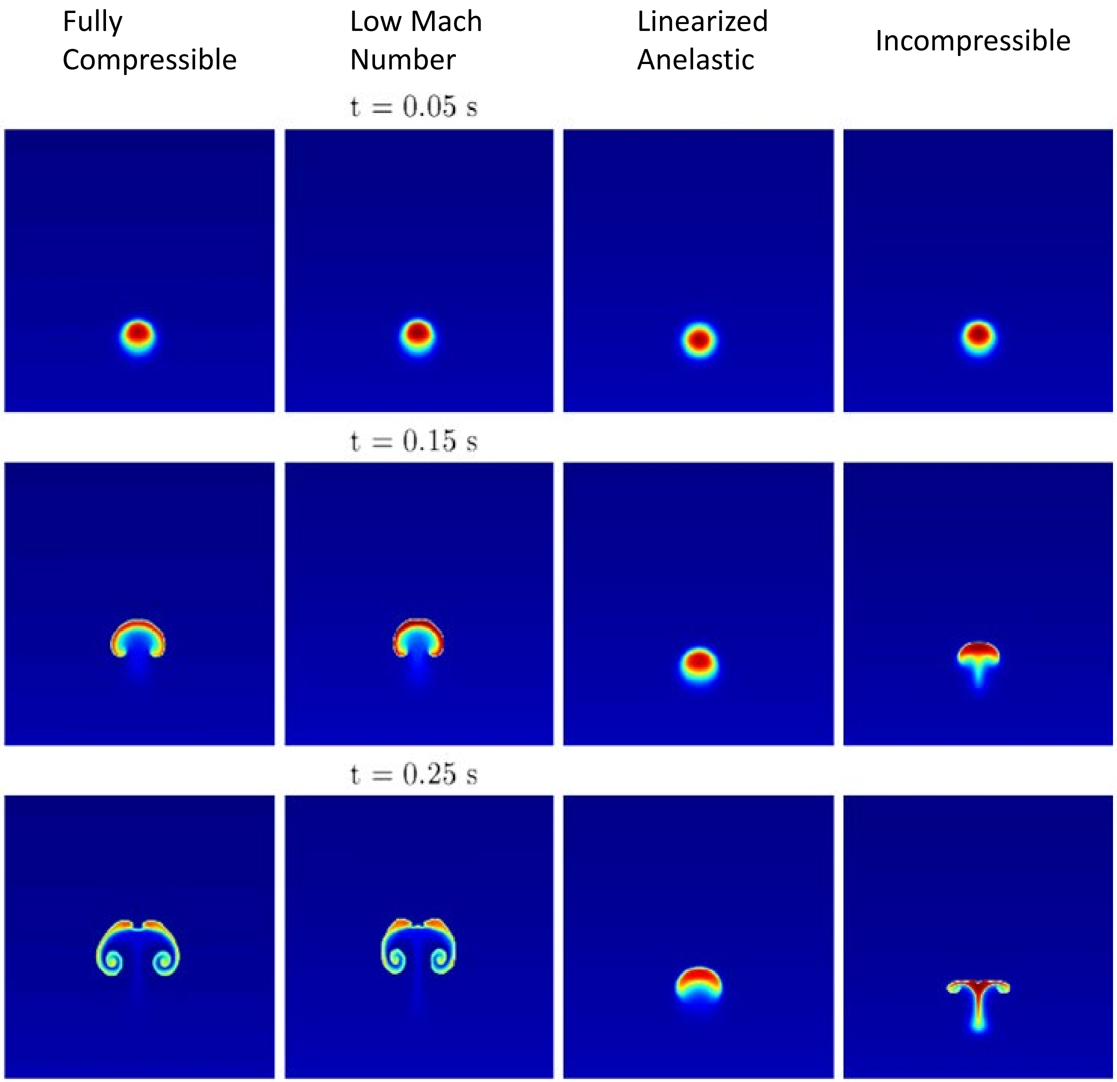
- **Pseudo-incompressible**
  - Local compressibility due to heat release
  - Compressibility due to stratification
  - Large thermodynamic perturbations ok

$$\nabla \cdot (p_0^{1/\gamma} U) = S$$

Assumes gamma-law gas

But none of these quite works …

Linearized Anelastic and Incompresible Don’t Work For Large Perturbations



So we made a new model …

Low Mach Number astrophysical modeling required a **new** model that allowed for buoyancy, time-evolving background stratification, reactions, and a non-ideal equation of state – but did not allow for acoustic waves

All previous models made too many assumptions.

$$\frac{\partial(\rho X_k)}{\partial t} = -\nabla \cdot (U \rho X_k) + \rho \dot{\omega}_k \ ,$$

Conservation of mass

$$\frac{\partial(\rho h)}{\partial t} = -\nabla \cdot (U \rho h) + \frac{D p_0}{D t} - \sum_k \rho q_k \dot{\omega}_k \ ,$$

Energy equation

$$\frac{\partial U}{\partial t} = -U \cdot \nabla U - \frac{\beta_0}{\rho} \nabla \left(\frac{p'}{\beta_0} \right) - \frac{(\rho - \rho_0)}{\rho} g e_r \ ,$$

Conservation of momentum

$$\nabla \cdot (\beta_0 U) = \beta_0 \left(S - \frac{1}{\overline{\Gamma}_1 p_0} \frac{\partial p_0}{\partial t} \right)$$

Constraint on velocity

where

$$S = -\sigma \sum_k \xi_k \dot{\omega}_k + \frac{1}{\rho p_\rho} \sum_k p_{X_k} \dot{\omega}_k$$

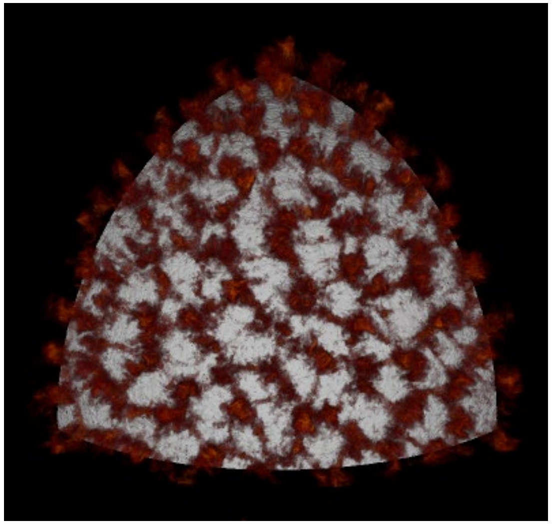
Use average heating to evolve base state.

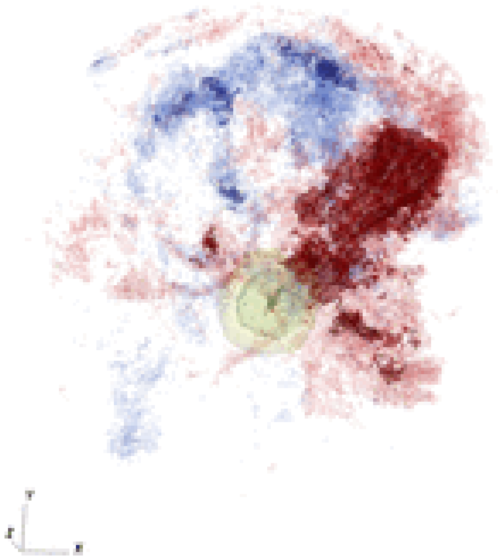
$$\frac{\partial p_0}{\partial t} = -w_0 \frac{\partial p_0}{\partial r} \quad \text{where} \quad w_0(r,t) = \int_{r_0}^{r} \overline{S}(r',t)\, dr'$$

MAESTRO Science (stay tuned…)

MAESTRO has been used to do new science:

- two types of Type Ia progenitors
    - Chandrasekhar mass
    - sub-Chandra
- X-ray bursts,
- convection in massive stars.

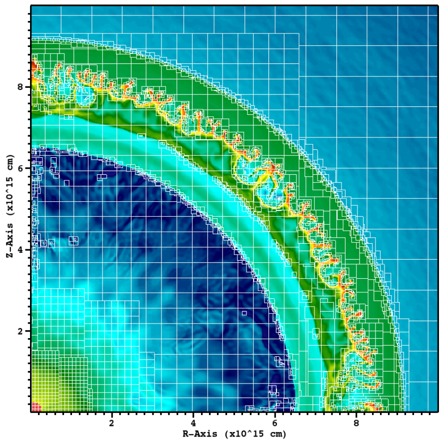


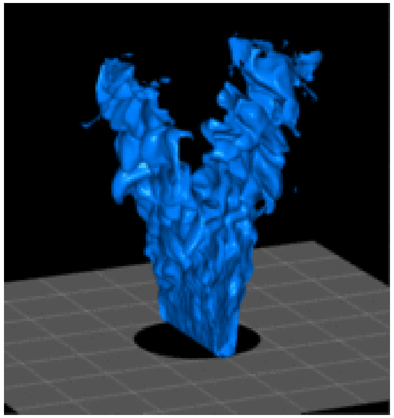


If you can solve one …

If we know how to solve other low Mach number flows – which we do! -- we can re-use most of the existing software. Elements of MAESTRO include:

- Projection method formulation
- Variable density Poisson solve for updating (perturbational) pressure and velocity
- ODE integrator for reaction network
- Adaptive mesh refinement – to address the range of spatial scales

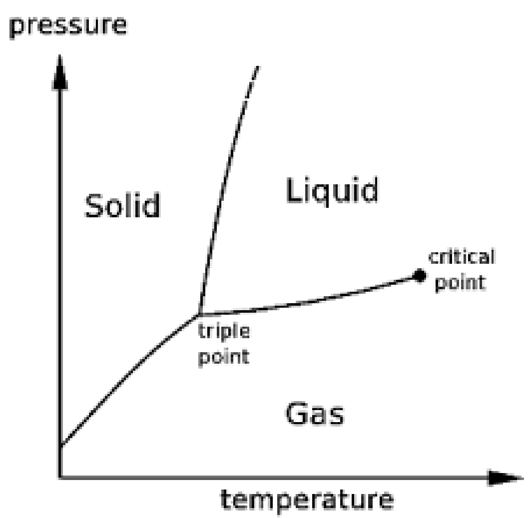


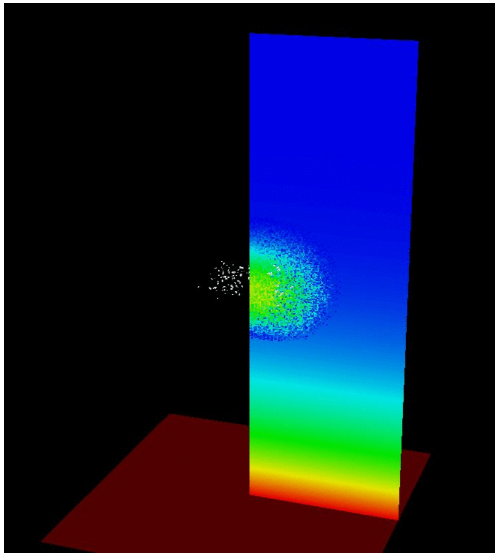


Moist atmospheric Flows

We can model moist atmospheric flows with this approach using a variation of MAESTRO

- Complication:
  - Clausius-Clapeyron gives equilibrium relationship for mixture of dry air, water vapor and liquid water
  - Divergence constraint needs heat release which is based on evaporation rate





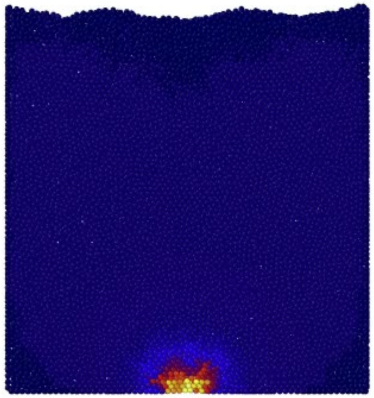
Multiphase Flows

Another interesting low Mach number flow – with interesting convective features – is multiphase flow in which solid particles occupy a volume fraction and exchange momentum with the ambient gas via drag terms.

$$\frac{\partial}{\partial t}(\varepsilon_g \rho_g) + \boldsymbol{\nabla} \cdot (\varepsilon_g \rho_g \boldsymbol{u}_g) = 0 \ ,$$

$$\frac{\partial}{\partial t}(\varepsilon_g \rho_g \boldsymbol{u}_g) + \boldsymbol{\nabla} \cdot (\varepsilon_g \rho_g \boldsymbol{u}_g \boldsymbol{u}_g) = -\varepsilon_g \boldsymbol{\nabla} p_g + \boldsymbol{\nabla} \cdot \boldsymbol{T}_g + \varepsilon_g \rho_g \boldsymbol{g} - \sum_{m=1}^{M} \boldsymbol{I}_{gm} \ .$$

where ε_g represents the volume fraction of gas



Low Mach Number Modeling is a Trade-Off

We have traded taking “too many” compressible time steps for having to solve a linear system for the pressure every “large” timestep.

There are other approaches -- in particular implicit time-stepping of the original equations – but re-formulating the governing equations rather than just re-formulating how we solve the original equations allows us different design choices.

It’s good to have options …

Thank you!

And I look forward to many interesting discussions